Laguerre Polynomial

Problem sheet

Problem 1.

Express $10 - 23x + 10x^2 - x^3$ in terms of Laguerre polynomials.

Problem 2.

Prove that $L_n(2x) = n! \sum_{n=0}^{\infty} \frac{(-1)^m 2^{n-m}}{m!(n-m)!} L_{n-m}(x)$.

Problem 3.

If $e^{-x} = \sum_{n=0}^{\infty} C_n L_n(x)$,then show that $C_n = \frac{1}{(2^{n+1} n!)}$.

Problem 4.

Show that $L_n(x)$ defined by $\exp(\frac{-xt}{1-t}) = (1-t) \sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!}$ satisfies diffrential equation $xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$.

Problem 5.

Laguerre polynomial $L_q(x)$ is defined by $e^{\frac{-xs}{1-s}} = \sum_{q=0}^{\infty} \frac{L_q(x)}{q!} s^q, \quad s < 1$. Show that $L_q' = qL_{q-1}' - qL_{q-1}$ and $L_{q+1} = (2q+1-x)L_q - q^2 L_{q-1}$.

Problem 6. State and prove that generating function for Laguerre polynomial.

Problem 7. Show that

(a) $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{\frac{-1}{2}}(x^2)$

(b) $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2)$

Problem 8. Derive the result

$$L_n^a(x) = \sum_{k=0}^{n} {}_2F_3\left[\begin{matrix} \frac{-1}{2}(n-k), \frac{-1}{2(n-k-1);} \\ \frac{3}{2}+k, \frac{1}{2}(1+\alpha+k), \frac{1}{2}(2+\alpha+k); \end{matrix} \middle| \frac{1}{4}\right] \frac{(-1)^k(1+\alpha)_n(2k+1)P_k(x)}{2^k(n-k)!(\frac{3}{2})_k(1+\alpha)_k}$$

Problem 9. Derive the result

$$L_n^a(x) = \sum_{k=0}^{n} {}_2F_2\left[\begin{matrix} \frac{-1}{2}(n-k), \frac{-1}{2(n-k-1);} \\ \frac{1}{2}(1+\alpha+k), \frac{1}{2}(2+\alpha+k); \end{matrix} \middle| \frac{1}{4}\right] \frac{(-1)^k(1+\alpha)_nH_k(x)}{2^k(n-k)!k!(1+\alpha)_k}$$

Problem 10. Show that

$$\int_0^t \frac{H_{2n}(\sqrt{x(t-x)})dx}{\sqrt{x(t-x)}} = (-1)^n\pi 2^{2n}\Big(\frac{1}{2}\Big)_n L_n\Big(\frac{1}{4}t^2\Big).$$

Problem 11. Show that if m is a non negative integer and α is not a negative integer,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{(1+\frac{1}{2}\alpha+\frac{1}{2}m)_n} \sum_{k=0}^{n} \frac{(\frac{1}{2}\alpha-\frac{1}{2}m)L_k^{\alpha}(-x)L_{n-k}^{m}(x)}{(1+\alpha)_k.}$$

Problem 12. Show that if m is a non negative integer and α is not a negative integer,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n(1+\alpha)_m}{(1+\alpha)_{m+n}} \sum_{k=0}^{n} \frac{(-m)_kL_k^{\alpha}(-x)L_{n-k}^{\alpha+2m}(x)}{(1+\alpha)_k.}$$

Problem 13. Use integration by parts, show that

$$\int_x^\infty e^{-y}L_n^\alpha(y)dy = e^{-x}[L_n^\alpha(x) - L_{n-1}^\alpha(x)].$$

Problem 14. Show that

$$\int_0^t x^\alpha(t-x)^{\beta-1}L_n^\alpha(x)dx = \frac{\Gamma(1+\alpha)\Gamma\beta}{\Gamma(1+\alpha+\beta)}.\frac{(1+\alpha)_nt^{\alpha+\beta}}{(1+\alpha+\beta)_n}L_n^{\alpha+\beta}(t).$$

Problem 15. Show that the Laplace transform of $L_n(t)$ is

$$\int_0^\infty e^{-st}L_n(t)dt = \frac{1}{s}\Big(1-\frac{1}{s}\Big)^n.$$

Problem 16. Show by the convolution theorem for Laplace transforms, or otherwise, that

$$L_n^t(t-x)L_m(x)dx = \int_0^t L_{m+n}(t) - L_{m+n+1}(t).$$

Problem 17. Evaluate the integral $\int_0^\infty x^\alpha e^{-x}[L_n^{(\alpha)}(x)]^2 dx$ and show that

$$\sum_{n=0}^\infty \int_0^\infty x^\alpha e^{-x}[L_n^\alpha(x)]^2 dx t^{2n} = (1-t)^{-2-2\alpha} \int_0^\infty x^\alpha exp[\frac{-x(1+t)}{1-t}]dx$$

$$= (1-t^2)^{-1-\alpha}\Gamma(1+\alpha)$$

$$= \sum_{n=0}^\infty \frac{\Gamma(1+\alpha+n)t^{2n}}{n!}.$$