Hermite Polynomial

Problem sheet

Problem 1.

Show that $H_n(x) = 2^{n+1}e^{x^2}\int_x^\infty e^{-t^2}t^{n+1}P_n(\frac{x}{t})dt$

Problem 2.

If $f(x)$ is a polynomial of degree m, show that $f(x)$ may be expressed in the form $f(x) = \sum_{r=0}^{m} c_r H_r(x)$,where,$C_r = \frac{1}{2^r r!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}f(x)H_r(x)dx$.

Deduce that $\int_{-\infty}^{\infty} e^{-x^2}fxH_n(x) = 0$.if $f(x)$ is a polynomial of degree less than n.

Problem 3.

Using the Generating function for Hermite polynomials, evaluate the values of

(i)$H_0(x)$ (ii)$H_1(x)$ (iii)$H_2(x)$ (iv)$H_3(x)$.

Problem 4.

Show that$H_n(x)$defined by $e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$ satisfies the diffrential equation $H_n''(x) - 2xH_n'(x) + 2xH_n(x) = 0$.

Problem 5.

The Hermite polynomial is defined for integral values of x by the identity $e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$.

Show that $H_n(x)$ satisfies the diffrential equation $H_n''(x) - 2xH_n'(x) + 2xH_n(x) = 0$. and $H_n(x)$ is given by $H_n(x) = (-1)^n e^{x^2}\frac{d}{dx}e^{-x^2}$.

Further show that $\int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2}dx = \sqrt{\pi}2^n n!$.

Problem 6.

Show that $\int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2} dx = \sqrt{\pi}2^n n!$ $=2^n n! \int_{-\infty}^{\infty} e^{-x^2}$

problem 7.

Show that

$(a)H_5(x) = 32x^5 - 160x^3 + 120x.$

$(b)H_6(x) = 64x^6 - 480x^4 + 720X^2 - 120.$

Problem 8.

show that for $n = 0, 1, 2, 3, ...$

(i)$H_{2n}(x) = \frac{2^{n+1}(-1)^n e^{x^2}}{\sqrt{\pi}} \int_0^{\infty} e^{-t^3} t^{2n} cos2xtdt.$

(ii)$H_{2n}(x) = \frac{2^{2n+2}(-1)^n e^{x^2}}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} t^{2n+1} sin2xtdt.$

Problem 9.

Prove that $H_n(-x) = (-1)^n H_n(x)$.

Problem 10.

Prove that $x^n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n! H_{n-2k}(x)}{2^n k!(n-2k)!}$ and express x^6 in terms of hermite polynomials.

Problem 11.

Use the fact that $\exp 2xt - t^2 = \exp 2xt - x^2t^2 \exp[t^2(x^2+1)]$to obtain the expansion

$$H_n(x) = \sum_{k=0}^{\frac{n}{2}} \frac{n! H_{n-2k}(1) x^(n-2k)(x^2+1)^k}{k!(n-2k)!}$$

Problem 12.

Use the expansion of x^n in a series of hermite polynomials to show that

$$\int_{-\infty}^{\infty} \exp(-x^2) x^n H_{n-2k}(x) dx = 2^{-2k} n! \frac{\sqrt{\pi}}{k!}$$

Note in perticular the special case k=0.

Problem 13.

Obtain the result

$$\int_{-\infty}^{\infty} \exp(-x^2) H_{2k}(x) H_{2s+1}(x) dx = \frac{(-1)^{k+s} 2^{2k+2s} (\frac{1}{2})k(\frac{3}{2}s)}{2s+1-2k}$$

Problem 14.

Show that

$$P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^{\infty} \exp(-t^2) t^n H_n(xt) dt,$$

Problem 15.

Use the Rodrigues formula $\exp(-x^2)H_n(x) = (-1)^n D^n \exp(-x^2)$ and iterated integration by parts to show that

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x) dx &= 0, \qquad && m \neq n \\ &= 2^n n! \sqrt{\pi}, \qquad && m = n \end{aligned}$$