Principle of Virtual work -

A virtual displacement, denoted by $\delta \vec{r_i}$ refers to an imaginary, infinitesimal instantaneous displacement of the co-ordinate that is consistent with the constraints. It is different from an actual displacement $d\vec{r_i}$ of the system occuring into a time interval dt.

It is called the virtual displacement is instantaneous.

As there is no actual motion of the system; the work done by the forces of constraint in such a virtual displacement is zero.

- Consider a scleronomic system of N particles (System independent of time) in equilibrium.

Let $\vec{F_i}$ be the force acting on the i^{th} particle. The force $\vec{F_i}$ is a vector addition of the externally applied force $\vec{F_i}^{e}$ and the forces of constraints $\vec{f_i}$ then -

$$\vec{F_i} = \vec{F_i}^e + \vec{f_i} \qquad — (1)$$

If $\delta\vec{r_i}$ is a virtual displacement of the ith particle, the virtual work done δW_i on the ith particle is given by -

$$\delta W_i = \vec{F_i} \cdot \delta\vec{r_i} \qquad — (2)$$

Now if the system is in equilibrium, the total force on each particle must be zero. $\vec{F_i} = 0$ for all i.

Hence the dot product $\vec{F_i} \cdot \delta\vec{r_i}$ is also zero.

so

$$\delta W_i = (\vec{F_i}^e + \vec{f_i}) \cdot \delta\vec{r_i} = 0 \qquad — (3)$$

$(i = 1, 2 \ldots N)$

The total virtual work done on the system δW is the sum of the above vanishing products. —

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} (\vec{F}_i^e + \vec{f}_i) \cdot \delta \vec{r}_i = 0$$

$$\delta W = \sum_{i=1}^{N} \vec{F}_i^e \cdot \delta \vec{r}_i + \underline{\sum_{i=1}^{N} \vec{f}_i \cdot \delta \vec{r}_i} = 0 \quad - (4).$$

under a virtual displacement, the work done by the forces of constraints is zero. This is valid for rigid bodies and most of the constraints that commonly occur. Therefore eqn (4) reduces to –

$$\delta W = \sum_{i=1}^{N} \vec{F}_i^e \cdot \delta \vec{r}_i = 0 \quad - (5)$$

which is the principle of virtual work and is stated as-

"In a N- particle system, the total work done by the external forces when virtual displacements are made is called virtual work and the total virtual work done is zero."

The coefficients $\delta \vec{r}_i$ in eqn (5) can no longer be set equal to zero as they are not independent. It should be noted that the principle of virtual work deals only with statics.

D' Alembert's principle ⇒

The principle of virtual work deals only with statics and the general motion of the system is not relevant here.

A principle that involves the general motion of the system was suggested by D' Alembert.

Consider the motion of an N-particle system.
Let the force acting on the i^{th} particle be $\vec{F_i}$.

By Newton's law –

$$\vec{F_i} = \dot{\vec{P_i}}$$

or $$\vec{F_i} - \dot{\vec{P_i}} = 0 \quad \text{— (1)}$$

It means that the i^{th} particle in the system will be in equilibrium under a force equal to the actual force plus a reverse effective force $-\dot{\vec{P_i}}$ as named by D' Alembert.

Then dynamics reduces to statics.
To this equivalent static problem, give a virtual displacement

$\delta\vec{r}_i$ which leads to –

$$\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \quad — (2)$$

or $$\sum_{i=1}^{N} (\vec{F}_i^e + \vec{f}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \quad — (3)$$

Restricting to conditions where the virtual work done by forces of constraints is zero.

$$\sum_{i=1}^{N} (\vec{F}_i^e - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \quad — (4)$$

This is D' Alembert's principle.