- Iteration Method -

Q.1 find a real root of the equation $x^3+x^2-1=0$ by using iteration method.

Sol - Let $\phi(x) = x^3+x^2-1$

$\phi(0) = -1 \quad (<0)$

$\phi(1) = 1+1-1 = 1 \quad (>0$

Hence the root of the equation lies between 0 & 1.

we write the given equation as -

$$x^2(x+1) = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{x+1}$$

$$x = \frac{1}{\sqrt{x+1}}$$

$$x = f(x)$$

where $f(x) = \frac{1}{\sqrt{x+1}}$

Now $f'(x) = \frac{1}{2(x+1)^{3/2}}$

for $x=0$, $|f'(x)| = \frac{1}{2} < 1$

for $x<1$, $|f'(x)| = \left|\frac{1}{2(x+1)^{3/2}}\right| < 1$

That is $|f'(x)| < 1$ for all x in $(0,1)$

Hence iterative method is applicable.

starting with $x_0 = 0.5$; we get -

$$x_1 = f(x_0) = \frac{1}{\sqrt{0.5+1}} = \frac{1}{\sqrt{1.5}} = \frac{\sqrt{1.5}}{1.5} \Rightarrow$$

$$\Rightarrow \quad \frac{1.224745}{1.5} = 0.81649$$

$$x_2 = f(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{1.81649}}$$

$$= \frac{\sqrt{1.81649}}{1.81649} = \frac{1.347776}{1.8165} = 0.74196$$

$$x_3 = f(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{1.74196}} = \frac{\sqrt{1.74196}}{1.74196}$$

$$= \frac{1.3198333}{1.74196} = 0.75767$$

$$x_4 = f(x_4) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{1.75767}} = \frac{\sqrt{1.75767}}{1.75767}$$

$$= \frac{1.3257715}{1.75767} = 0.75428$$

$$x_5 = f(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{1.75428}} = \frac{\sqrt{1.75428}}{1.75428}$$

$$= \frac{1.3244923}{1.75428} = 0.75501$$

$$x_6 = f(x_5) = \frac{1}{\sqrt{1.75501}} = 0.75485$$

$$x_7 = f(x_6) = \frac{1}{\sqrt{1.75485}} = 0.75488$$

matched

$$x_8 = f(x_7) = \frac{1}{\sqrt{1.75488}} = 0.75488$$

Hence correct to 3 places of decimals, the root of the given equation is 0.755

. Solve the equation $3x - \cos x - 1 = 0$ by applying iterative method.

Sol. we have -

$$f(x) = 3x - \cos x - 1$$

$$f(0) = -1 - 1 = -2 \quad (<0)$$

$$f(\pi/2) = \frac{3\pi}{2} - 0 - 1 = +ve \quad (>0)$$

Hence a root lies between $0 - \pi/2$.

we can rewrite the given equation -

$$3x - \cos x - 1 = 0$$

$$3x = 1 + \cos x$$

$$x = \frac{1}{3}(1 + \cos x) = \phi(x)$$

Then we have -

$$\phi'(x) = -\frac{\sin x}{3}$$

such that -

$$|\phi'(x)| = \frac{1}{3}|\sin x| < 1$$

for all x and in particular in $(0, \pi/2)$

Then the iteration method can be applied. starting with $x_0 = 0$

the succesive approximations are as follows -

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos 0 + 1) = \frac{2}{3} = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos(0.6667) + 1)$$

$$x_2 = \frac{1}{3}(0.7859 + 1) = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.5953) + 1]$$

$$x_3 = \frac{1}{3}[0.8281 + 1] = 0.6093$$

$$x_4 = \phi(x_3)$$

$$= \frac{1}{3}[\cos(0.6093)+1]$$

$$= \frac{1}{3}[0.8202+1]$$

$$x_4 = 0.6067$$

$$x_5 = \phi(x_4)$$

$$= \frac{1}{3}[\cos(0.6067)+1]$$

$$= \frac{1}{3}[0.8214+1]$$

$$= 0.6071$$

$$x_6 = \phi(x_5)$$

$$= \frac{1}{3}[\cos(0.6071)+1]$$

$$= \frac{1}{3}[0.8213+1]$$

$$x_6 = 0.6071$$

Since x_5 and x_6 are almost same, hence the root is 0.607 correct to 3 places of decimals.

Q.3. Find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using iteration method.

Sol. $f(x) = 2x - \log_{10} x - 7 = 0$

$$f(3) = 6 - \log_{10} 3 - 7 = 6 - 0.4771 - 7$$

$$= -1.04771 \quad (<0)$$

and $f(4) = 8 - \log_{10} 4 - 7 = 1 - 0.602 = 0.398 \ (>0)$

Thus a root lies between 3 and 4.

we can write the given equation as:

$$x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$$

Then $\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right).$

Evidently $|\phi'(x)| < 1$. when $3 < x < 4$ $(\log_e 10 = 0.4343)$

Now $|f(4)| < |f(3)|$

Therefore the root is near to 4.

we now apply iteration method starting from

$x_0 = 3.6$

The successive approximations are computed as -

$$x_1 = \phi(x_0) = \frac{1}{2}[\log_{10}(3.6) + 7]$$

$$= \frac{1}{2}[0.5563 + 7] = \frac{7.5563}{2} = 3.7781$$

$$x_2 = \phi(x_1) = \frac{1}{2}[\log_{10}(3.7781) + 7]$$

$$= \frac{1}{2}[0.5773 + 7]$$

$$= \frac{7.5773}{2}$$

$$x_2 = 3.7886$$

$$x_3 = \frac{1}{2}[\log_{10}(3.7886) + 7]$$

$$= \frac{1}{2}[0.5785 + 7] = \frac{7.5785}{2} = 3.7892$$

$$x_4 = \phi(x_3)$$

$$= \frac{1}{2}\left[\log(3.7892) + 7\right]$$

$$= \frac{1}{2}\left[0.5785 + 7\right]$$

$$x_4 = \frac{7.5785}{2} = 3.7892$$

Since x_3 and x_4 are almost equal (upto 4 places of decimals), we conclude that root is $\underline{\underline{3.7892}}$.