Lecture - 2

Errors and Their Computations

There are Two Kind of numbers.

* Exact Numbers ⇒ $1, 2, 3, \ldots \frac{1}{2}, 3/2, \sqrt{2}, \pi, e$ etc.

* Approximate Numbers ⇒ They represent the numbers to a certain degree of accuracy.

Thus an approximate value of $\pi \Rightarrow 3.1416$

better approx. $\Rightarrow 3.14159265$

But we can't write exact value of π.

* Significant figures/digits ⇒ The digits used to express numbers.

Thus the numbers 3.1416 or 4.0687 contain five significant digits each.

In numerical computations, we come across numbers which have large number of digits and it will be necessary to cut them to a usable numbers of figures. This process is called rounding-off

To round-off a number to n significant digits, discard all digits to the right of the n^{th} digit and if this discarded number is -

a) less than half a unit in the n^{th} place, leave the n^{th} digit unaltered.

b) greater than half a unit in the n^{th} place, increase the n^{th} digit by unity.

c) exactly half a unit in the n^{th} place, increase the n^{th} digit by unity if its odd; otherwise leave it unchanged.

The number thus rounded-off is said to be correct to n significant figures.

Examples :- 1.6583 to 1.658

3.14159 to 3.142

In hand computations, the round-off error can be reduced by carrying out the computations to more significant figures at each step of the computation.

Absolute, Relative and percentage errors :—

Absolute error is the numerical difference between the true value of a quantity and its approximate value. Thus, if X is the true value of a quantity and X_1 is its approximate value, then the absolute error E_A is given by –

$$E_A = X - X_1 = \delta x$$

The relative error E_R is defined by –

$$E_R = \frac{E_A}{X} = \frac{\delta X}{X}.$$

and the percentage error by –

$$E_p = 100 \, E_R.$$

Let ΔX be a number such that -

$$|X_1 - X| \leq \Delta X$$

then Δx is an upper limit on the magnitude of the absolute error and is said to measure absolute accuracy.

Similarily.

$$\frac{\Delta x}{|x|} \simeq \frac{\Delta x}{|x_1|}$$

measures relative accuracy.

General Rules :—

when Two number are added or subtracted — then the magnitude of the absolute error in the result is the sum of the magnitudes of the absolute errors in the two numbers.

More generally, if $E_A^1, E_A^2 \ldots E_A^n$ are the absolute errors in n numbers., then the magnitude of the absolute error in the sum is given by-

$$|E_A^1| + |E_A^2| + \ldots |E_A^n|$$

Absolute Error in a product of two numbers a & b.

we write $E_A = (a+E_A^1)(b+E_A^2) - ab$

where E_A^1 and E_A^2 are the absolute errors in a and b respectively.

Thus.

$$E_A = aE_A^2 + bE_A^1 + E_A^1 E_A^2$$

$$= bE_A^1 + aE_A^2 \quad (\text{approx.}).$$

Absolute error in the quotient a/b is given as–

$$\Rightarrow \quad \frac{a+E_A^1}{b+E_A^2} - \frac{a}{b} = \frac{bE_A^1 - aE_A^2}{b(b+E_A^2)}$$

$$\Rightarrow \quad \frac{bE_A^1 - aE_A^2}{b^2(1+E_A^2/b)}$$

$$\Rightarrow \quad \frac{bE_A^1 - aE_A^2}{b^2}$$ (assuming E_A^2/b is small in comparison with 1)

$$\Rightarrow \frac{a}{b}\left[\frac{E_A^1}{a} - \frac{E_A^2}{b}\right]$$

PROBLEMS

Q.1 An approximate value of π is given by $x_1 = \frac{22}{7}$ $= 3.1428571$ and its true value is $x = 3.1415926$ find the absolute and relative errors.

Solution – we have.

$$E_A = x - x_1$$

$$E_A = -0.0012645$$

$$E_R = \frac{-0.0012645}{3.1415926}$$

$$E_R = -0.000402$$

Q.2. Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to 4 significant digits and find its absolute and relative errors.

Sol. we have $\sqrt{3} + \sqrt{5} + \sqrt{7} = 1.732 + 2.236 + 2.646$

$$= 6.614.$$

$$E_A = 0.0005 + 0.0005 + 0.0005 = 0.0015$$

The total absolute error shows that sum is correct to 3 significant figures only. Hence we take $S = 6.61$

$$E_R = \frac{0.0015}{6.61} = \underline{\underline{0.0002}}$$