Lecture-12

Factorization Method

LU Decomposition method / Triangular method

Ques: Solve the Equations –

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2z=8$$

by the factorization method.

Sol. Here the coeficient matrix is A.

$$A=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

Let $A=LU$

where

$$L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}$$

$$U=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

Then we have –

$$L\times U=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Row 1.

$u_{11}=2$ —(1)

$u_{12}=3$ —(2)

$u_{13}=1$ —(3)

Row. 2

$$l_{21} U_{11} = 1 \quad -(4)$$

$$l_{21} U_{12} + U_{22} = 2 \quad -(5)$$

$$l_{21} U_{13} + U_{23} = 3 \quad -(6)$$

Row. 3

$$l_{31} U_{11} = 3 \quad -(7)$$

$$l_{31} U_{12} + l_{32} U_{22} = 1 \quad -(8)$$

$$l_{31} U_{13} + l_{32} U_{23} + U_{33} = 2 \quad -(9)$$

from (4) $\quad l_{21} \times 2 = 1$

$$l_{21} = 1/2$$

from (5) $\quad \frac{1}{2} \times 3 + U_{22} = 2$

$$U_{22} = 2 - \frac{3}{2}$$

$$U_{22} = \frac{1}{2}$$

from (6)

$$\frac{1}{2} \times 1 + U_{23} = 3$$

$$U_{23} = 3 - \frac{1}{2}$$

$$U_{23} = 5/2$$

from (7) $\quad l_{31} \times 2 = 3$

$$l_{31} = 3/2$$

from (8) $\quad \frac{3}{2} \times 3 + l_{32} \times \frac{1}{2} = 1$

$$\frac{l_{32}}{2} = 1 - \frac{9}{2} = -7/2 \Rightarrow l_{32} = -7$$

from ⑨

$$\cancel{\frac{3}{2}\times 1 + U_{23} = 3}$$

$$\cancel{U_{23} = 3 - \frac{3}{2}}$$

$$\cancel{U_{23} = 3/}$$

$$\frac{3}{2}\times 1 + (-7)\frac{5}{2} + U_{33} = 2$$

$$\frac{3}{2} - \frac{35}{2} + U_{33} = 2$$

$$U_{33} = 2 + \frac{32}{2}$$

$$U_{33} = \frac{36}{2}$$

$$U_{33} = 18$$

Thus we can write:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

and $AX = B$ $\qquad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Thus $LUX = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

putting $UX = Y$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad — (10)$$

where.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad — (11)$$

Now from eqn (10)

we get -

$$y_1 = 9$$

$$\frac{1}{2}y_1 + y_2 = 6 \Rightarrow \frac{9}{2} + y_2 = 6 \Rightarrow y_2 = 6 - \frac{9}{2} = \frac{3}{2}$$

and $\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \Rightarrow \frac{3}{2}(9) - 7 \times \frac{3}{2} + y_3 = 8 \Rightarrow y_3 = 8 \cancel{+} \frac{\cancel{6}}{2} = \underline{\underline{5}}$

Thus $y_1 = 9$

$y_2 = 3/2$

$y_3 = 5$

Thus.

Eq (11) becomes

$$\begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus $2x + 3y + z = 9 \quad — (12)$

$\frac{1}{2}y + \frac{5}{2}z = 3/2 \quad — (13)$

$18z = 5 \Rightarrow \quad — (14)$

$z = 5/18$

from (13) $\Rightarrow$ $\frac{y}{2} + \frac{5 \times 5}{2 \times 18} = \frac{3}{2}$

$\frac{y}{2} = \frac{3}{2} - \frac{25}{36}$

$\frac{y}{2} = \frac{54 - 25}{36}$

$\frac{y}{2} = \frac{29}{36}$

$y = \frac{29}{18}$

from (12) $2x + \frac{3 \times 29}{18} + \frac{5}{18} = 9$

$2x = -\frac{87}{18} - \frac{5}{18} + 9$

$2x = -\frac{92}{18} + 9$

$2x = \frac{-92 + 162}{18} = \frac{70}{18}$

$x = \frac{70}{18 \times 2} = \frac{35}{18}$

Thus $x = \frac{35}{18}$

$y = \frac{29}{18}$

$z = \frac{5}{18}$

Q. Solve the following. using factorization method.

$$x+y+z=9$$

$$2x-3y+4z=13$$

$$3x+4y+5z=40$$

Solution.

$x=1$

$y=3$

$z=5$

Q. Solve the following equations by factorization method.

$$10x-7y+3z+5u=6$$

$$-6x+8y-z-4u=5$$

$$3x+y+4z+11u=2$$

$$5x-9y-2z+4u=7$$

Soln.

$u=1$

$z=-7$

$y=4$

$x=5$