Linear Programming

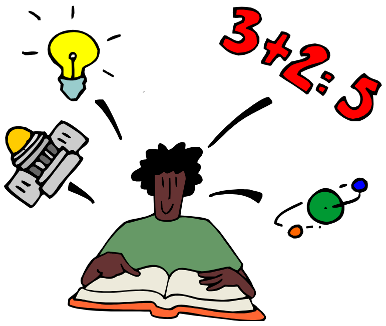
What is LP?

The word linear means the relationship which can be represented by a straight line .i.e the relation is of the form ax +by=c. In other words it is used to describe the relationship between two or more variables which are proportional to each other.

The word “programming” is concerned with the optimal allocation of limited resources.

INTRODUCTION TO LPP

- The most important function of management is decision making, A large number of decision



Definitions of LP

- LP is a mathematical modeling technique useful for the allocation of “scarce or limited “ resources, such as labor, material, machine ,time ,warehouse space ,etc…,to several competing activities such as product ,service ,job, new equipments, projects, etc...on the basis of a given criteria of optimality.
- The phase scare resource means resource that are not available in infinite quantity during the planning period.
- The criteria of optimality ,generally is either performance ,return on investment ,profit ,cost ,utility ,distance etc.

Definitions of LPP

A mathematical technique used to obtain an optimum solution in resource allocation problems, such as production planning.

It is a mathematical model or technique for efficient and effective utilization of limited recourses to achieve organization objectives (Maximize profits or Minimize cost).

General Structure of LP Model

- There are *n* variables in *m* constraints to be solved

$$Max / Min \quad Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$S.t.$$

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \le / \ge / = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \le / \ge / = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \le / \ge / = b_m$$

$$x_i \ge 0, \quad i = 1, 2, \ldots, n$$

Linear Programming - LP

- The maximization or minimization of some quantity is the objective in LP problems.
- All LP problems have constraints that limit the degree to which the objective can be pursued.
- A feasible solution satisfies all the constraints.
- An optimal solution is a feasible solution that results in the largest possible objective function value when maximizing (or smallest when minimizing).
- A graphical solution method can be used to solve a linear program with two variables.

Linear Programming - LP

- A function $f(x_1, x_2, \ldots, x_n)$ is a linear function *iff* for some set of constants $c_1, c_2, \ldots, c_n$, $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- For any linear function $f(x_1, x_2, \ldots, x_n)$ and any number b, the inequalities $f(x_1, x_2, \ldots, x_n) \leq b$ and $f(x_1, x_2, \ldots, x_n) \geq b$ are linear inequalities
- Solving the problem means finding the values of the decision variables that:
  - satisfy the constraints
  - optimize the objective function

- **The objective function:** The objective (goal) of each LPP is expressed in terms of decision variables to optimize the criterion of optimality (also called measure of performance) such as profit, cost ,revenue etc.. In its general form, it is represented as :

$$\textbf{Optimize (Max/ Min) } Z = C1x1 + C2x2 + \ldots\ldots.Cnxn$$

where Z is the measure of performance variable, which is a function of x1, x2,…..xn. Quantities c1,c2….cn are parameters that represent the contribution of unit of respective variables. The optimal value of given objective function is obtained by the graphical method or simplex method.

Decision variables: We need to evaluate various alternatives (course of action) for arriving at the optimal value of objective function. The evaluation of these alternatives is guided by the nature of objective function and availability of resource. For this we presume certain activities (decision variables) usually denoted by x1,x2….xn. The value of these activities represents the extent to which each of these is performed.

The constraints: There are always certain limitations (or constraints) on the use of resources e.g. labour, m/c, raw material, money etc.. That limit the degree to which an objective can be achieved. Such constraints must be expressed as linear equalities or inequalities in terms of decision variables. The soln of LP model must satisfy these constraints.

Assumptions Of LPP

- Certainty
- Additivity
- Linearity (or proportionality)
- Divisibility (or continuity)
- Finite choices

LP Assumptions

- *When we use LP as an approximate representation of a real-life situation, the following assumptions are inherent:*
  - *Proportionality. - The contribution of each decision variable to the objective or constraint is directly proportional to the value of the decision variable.*
  - *Additivity. - The contribution to the objective function or constraint for any variable is independent of the values of the other decision variables, and the terms can be added together sensibly.*
  - *Divisibility. - The decision variables are continuous and thus can take on fractional values.*
  - *Deterministic. - All the parameters (objective function coefficients, right-hand side coefficients, left-hand side, or technology, coefficients) are known with certainty.*

Optimization & Problem Solving

- Optimization attempts to express the goal of solving a problem in the best way.
  - Running business to maximize profit, minimize loss and risk
  - Design a bridge to maximize strength
  - Selecting a flight plan to minimize time and fuel use
- To solve a problem:
  - Define and formulate the problem
  - Build a mathematical model of the problem
  - Verify the model
  - Identify and evaluate suitable feasible alternatives
  - Select the best alternative
  - Interpret and present your decision

Methods of Solving Linear Programming Problems (LPP)

There are Two important methods of solving L.P.P. They are:

(1) Graphical method
(2) Simplex method

Extreme Points and the Optimal Solution

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.
  - Find constraint intersections
  - Start at, say point (1)
  - Substitute in Z equation
  - Move to point (2)
  - Then (3)… etc
  - Seek improvement
  - Find the corner point (X1, X2) for the best Z value– Optimal Solution

Advantages of LPP

- LP helps in attaining the optimum use of productive resources.
- LP improves the quality of decisions .The decision-making approach of the user of the technique becomes more objective and less subjective.
- It also helps in providing better tools for adjustments to meet changing conditions. Most business problems involve constraints like raw material availability, etc which must be taken into consideration. Just coz we can produce so many units does not mean that they can be sold. LP can handle such situation also since it allows modification of its mathematical solutions.
- Highlighting the bottle neck in the production process is the most significant advtg of this process.

Limitations of LPP

- LP treats all relationships among decision variables as linear. However ,generally, neither the OF nor the constraints in real life situations concerning business and industrial problems are linearly related to the variables .It may yield fractional valued ans for decision variables, round the same values may not yield an optimal solu.
- LP model does not take into consideration the effect of time and uncertainty.
- Parameters appearing in the model are assumed to be constant but in real life situations, they are frequently neither known nor constant.
- It deals only with single objectives, whereas in real-life situations we may come across conflicting multi-objective problems.
- For large problems having many limitations and constraints, the computational difficulties are enormous, even when the assistance of computer is available. For it, the main problem can be fragmented into several small problems and solving each one separately.

Formulation to LPP (Production allocation problem)

Case: 1

A manufacturer produces two models M_1 and M_2 product. Each unit of model M_1 requires 4 hours grinding and 2 hours polishing. Each unit of model M_2 requires 2 hours grinding and 5 hours of polishing. The manufacturer has 2 grindings each of which works for 40 hours per week. There are 3 polishing each of which works for 60 hours per week. Profit on model M_1 is Rs. 300 per unit and profit on model M_2 is 400 per unit. The manufacturer has to allocate his production capacity so as to maximize his profit.

→ Formulate the LPP.

Formulation to LPP
(Production allocation problem)

Solution:

Let X: Number of units of model M_1 to be produced.

Y: Number of units of model M_2 to be produced.

	Model M_1	Model M_2	Requirement
Number of units	X	Y	Maximize
Grinding time(hrs.)	4 X	2 Y	$\leq$ 80
Polishing time(hrs.)	2 X	5 Y	$\leq$ 180
Profit (Rs)	300 X	400 Y	

Conversion to LPP (Production allocation problem) (cont..d)

Maximize $Z = 300X + 400Y$ — Objective Function

Subject to (S.t) $4X + 2Y \leq 80$

$2X + 5Y \leq 180$ — Subjective Function

and $X \geq 0$, $Y \geq 0$ — Decision variable (non negative)

Graphical method problem

- Case-1 Linear Programming Problem

  Solve the following L.P.P. graphically.

  Maximize $Z = 2000X + 1000Y$

  Subject to $X + Y \leq 400$

  $8X + 5Y \leq 2600$

  and $X \geq 0, Y \geq 0$

Case-1 Solution to Linear Programming Problem

– Computation of corner point:

Equation	Putting X=0, then Y=?	Putting Y=0, then X=?	Corner Point(X,Y) on the line
X+Y=400 X+Y=400	Y=400 -----	---- X=400	(0,400) (400,0)
8X+5Y=2600 8X+5Y=2600	Y= 2600/5 = 520 -----	----- X=2600/8 =325	(0,520) (325,0)
X=0	Y axis		
Y=0	X axis		

Simplex Method

Summary:

- Algorithm
- Various steps in Simplex Method
- Definition of Slack, Surplus, Artificial Variables
- Two Artificial Variable Technique:

  ***The Big M-Method (OR method of penalties) , A.Charles**

  *** The two-phase method, Dantzig, Orden and Wolfe**

- **Condition of having unbounded soln**

Duality in LPP

Summary

- Definition
- Characteristics
- Dual problem when primal is in canonical form
- Dual problem when primal is in standard form
- Duality theorems
- Dual price/ shadow price
- Advantages of Duality

Duality in LPP

- In the context of LPP ,Duality implies that each LPP can be analysed in two ways but having equivalent solutions.
- The main focus of Dual is to find for each resource its best marginal value(dual price or shadow price)
- Shadow price= $\dfrac{\text{Change in optimal objective function}}{\text{Unit Change in the availability of resources}}$

Dual Problem

- A firm manufactures two products Aand B on m/cs I and II as shown below:

Machine	Product A	Product B	Available hrs
I	30	20	300
II	5	10	110
Profit per unit(Rs)	6	8	

The total time available is 300 hrs and 110 hrs on m/c I and II, respectively. Product A and B contributes Rs 6 and Rs 8 per unit, respectively. Determine the optimum product mix. Write the dual of this LPP and give its economic interpretation.

Sensitivity Analysis

- Sensitivity analysis is carried out after the optimal solution of an LPP is obtained. That's the reason ,it is also referred to as post-optimality analysis.

  The goal is to determine whether changes in the model's coefficients will live the optimal soln. unchanged, and if not, how a new optimal (assuming one exist) can be obtained efficiently.

Sensitivity Analysis

- In sensitivity analysis , we determine the range over which the LP model parameters can change without affecting the current optimal solution.

What If:

- **Change in coefficients of Variables in the Objective Function**
- **Change of the Coefficients of decision variables on the L.H.S. of the constraints (aij)**
- **Change of the RHS of constraints/ Resources**
- **Adding New Variables**
- **Adding New Constraints**

Sensitivity Analysis

- A company produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hour on machine 2. The revenue per unit of product 1 and 2 are Rs.30 and Rs.20 respectively. The total daily processing time for each machine is 8 hours.

Sensitivity Analysis

- Question1: If company can increase the capacity of both the machines then which machine should receive higher priority?
- Question2: A suggestion is made to increase the capacities of machine 1 and 2 at the additional cost of Rs.10 per hour. Is this advisable.
- Question3: If the capacity of machine 1 is increased from 8 hours to 13 hours, how will this increase impact the optimum revenue.
- Question4: Suppose that the unit revenues for product 1 and 2 are changed to Rs.35 and Rs.25 respectively. Will the current optimum remain the same.
- Question5: Suppose that the unit revenue of product 2 is fixed at its current value of C2=Rs.20. What is the associated range for C1, unit revenue for product 1 that will keep the optimum unchanged.