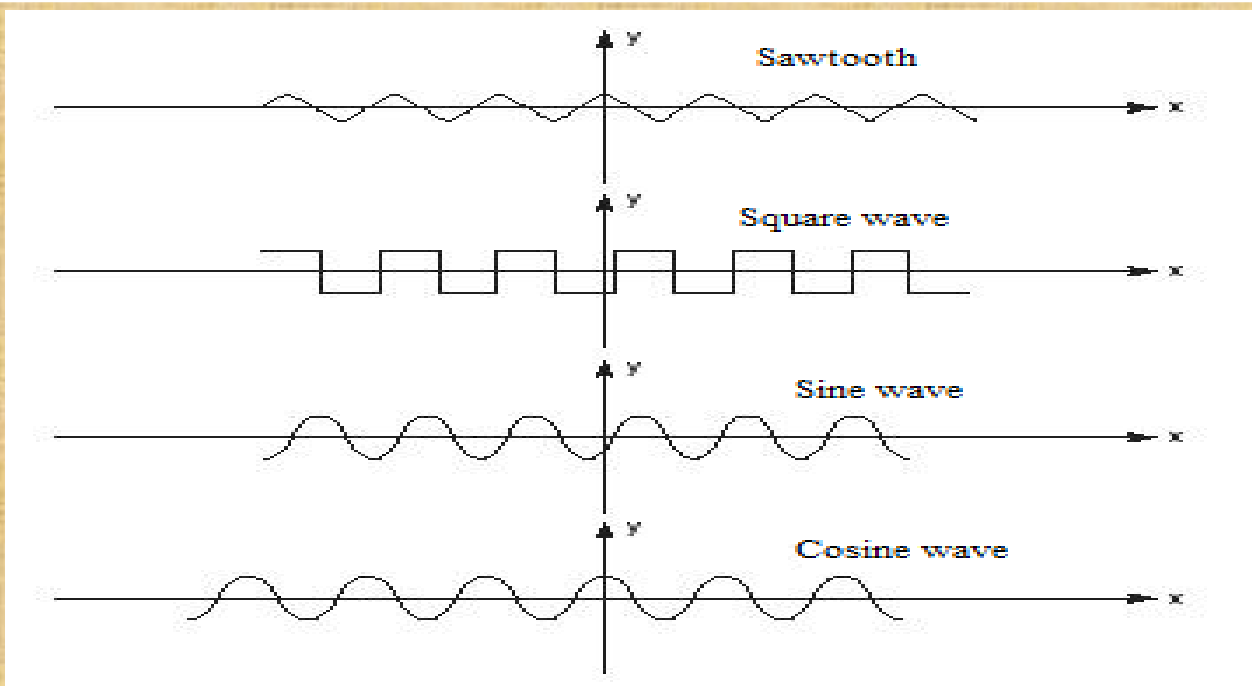
Fourier series

Lecture 4

Periodic function

A graph of periodic function $f(x)$ that has period L exhibits the same pattern every L units along the $x-$axis, so that

$f(x \pm L) = f(x)$ for every value of x.



Euler formulae

The fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{con} nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where : a_0, a_n and b_n are constant, known as fourier coefficient,

which can be evaluated by Euler formula

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx$$

Case.I : If $c = 0$ then fourier series in $0 < x < 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{ con } nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \;,\; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad \& \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Case.II : If $c = -\pi$ then fourier series in $-\pi < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{ con } nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \,,\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \& \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Example : Find the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

Let $f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{con} nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(1)

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

now $a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left(\frac{e^{-x}}{-1}\right)_0^{2\pi} = \frac{1}{\pi}(1 - e^{-2\pi})$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$

$\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx]$

$$a_n = \left[\frac{e^{-x}}{\pi(1+n^2)}(-\cos nx + n\sin nx)\right]_0^{2\pi} = \frac{1}{\pi(1+n^2)}[e^{-2\pi}(-\cos 2n\pi - (-1))]$$

$$\because \cos n\pi = (-1)^n, \sin n\pi = 0, \quad \cos(n+\frac{1}{2})\pi = 0, \quad \sin(n+\frac{1}{2})\pi = (-1)^n$$

$$a_n = \frac{1}{\pi(1+n^2)}(1-e^{-2\pi}) \;\;, \; n = 1, 2, 3, \ldots\ldots\ldots\ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx = \left[\frac{e^{-x}}{\pi(1+n^2)}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$b_n = \frac{1}{\pi(1+n^2)}[e^{-2\pi}(-n)-(-n)]$$

$$b_n = \frac{n}{\pi(1+n^2)}(1-e^{-2\pi}) \quad , \; n = 1, 2, 3, \ldots\ldots\ldots$$

Now substitute $a_{0,}$ a_n and b_n in equation (1) we get

$$\text{Thus } e^{-x} = \frac{1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right) + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\left\{\frac{1}{1+n^2}\cos nx + \frac{n}{1+n^2}\sin nx\right\}$$

$$\because \int UVdx = UV_1 - U^{/}V_2 + U^{//}V_3 - U^{///}V_4 + \ldots\ldots\ldots$$

$$U^{/} = \frac{dU}{dx}, U^{//} = \frac{d^2U}{dx^2}, U^{///} = \frac{d^3U}{dx^3} \;\&\; V_1 = \int Vdx\,, V_2 = \int\int Vdx^2\,, V_3 = \int\int\int Vdx^3 ..\text{etc}$$

Example : Obtain the fourier series for $f(x) = x - x^2$ in $-\pi < x < \pi$,

Hence prove $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

Solution : Let $f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{con} nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx \quad \{\text{sin ce x is an odd function}\}$$

$$a_0 = \frac{1}{\pi}\left[-\frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2}{3}\pi^2 \qquad \left\{\begin{array}{l}\text{Odd function if } f(-x) = -f(x) \\ \text{Even function if } f(-x) = f(x)\end{array}\right\}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nxdx$$

$$a_n = 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nxdx \quad \{\text{first int egral zero due an odd function}\}$$

$$a_n = -\frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\frac{-\cos nx}{n^2} + 2\frac{-\sin nx}{n^3}\right]_{-\pi}^{\pi} = \frac{-1}{\pi}\left[2\pi\frac{\text{con } n\pi}{n^2} + 2\pi\frac{\cos n\pi}{n^2}\right]$$

$$a_n = \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - 0$$

$$b_n = \frac{1}{\pi}\left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right] = \frac{-2(-1)^n}{n}$$

$$\text{Now } f(x) = x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2} \text{con } nx + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$f(x) = x - x^2$ is continuous at $x = 0$,

$$\therefore f(0) = 0 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2}$$

$$\text{i.e. } \frac{\pi^2}{3} = -4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\ldots\right]$$

$$\text{or } \frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \ldots\ldots\ldots\ldots\right]$$

Fourier series for Discontinues function

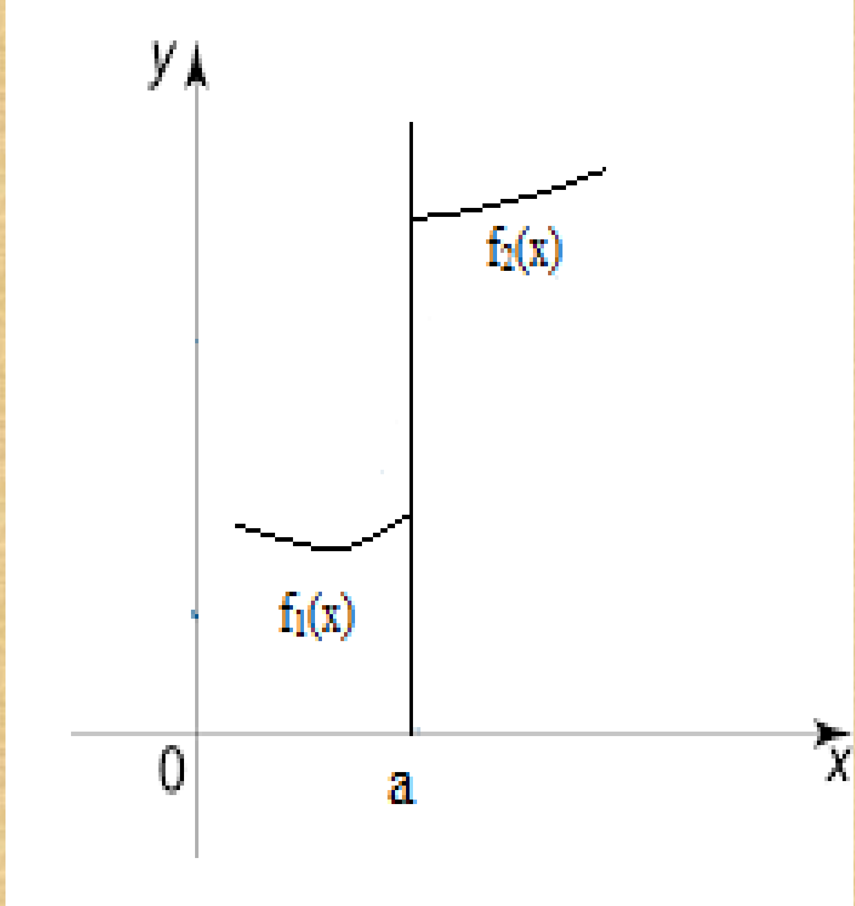
$$\text{If } f(x)=\begin{cases} f_1(x)\,, & c<x<a \\ f_2(x), & a<x<c+2\pi \end{cases}\quad, \quad i.e., a \text{ is the}$$

point of discontinuity, then

$$a_0=\frac{1}{\pi}\left[\int_c^a f_1(x)\,dx+\int_a^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n=\frac{1}{\pi}\left[\int_c^a f_1(x)\cos nx\,dx+\int_a^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$\&\, b_n=\frac{1}{\pi}\left[\int_c^a f_1(x)\sin nx\,dx+\int_a^{c+2\pi} f_2(x)\sin nx\,dx\right]$$



$$\text{at the point of discontinuity } x=a,\ f(x)=\frac{1}{2}[f(a-0)+f(a+0)].$$

Example : Obtain the fourier series for $f(x)=\begin{cases}-1 & -\pi<x<0\\ 1 & 0<x<\pi\end{cases}$

Solution : Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \text{ con } nx+\sum_{n=1}^{\infty}b_n \sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}-1\,dx+\frac{1}{\pi}\int_{0}^{\pi}1\,dx=\frac{1}{\pi}[(-x)_{-\pi}^{0}+(x)_0^{\pi}]=\frac{1}{\pi}[-\pi+\pi]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}-\cos nx dx+\frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx=\frac{1}{\pi}[(-\frac{\sin nx}{n})_{-\pi}^{0}+(\frac{\sin nx}{n})_0^{\pi}]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}-\sin nx dx+\frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx=\frac{1}{\pi}[(\frac{\cos nx}{n})_{-\pi}^{0}+(-\frac{\cos nx}{n})_0^{\pi}]$$

$$b_n = \frac{1}{n\pi}\left[\{1-\cos(-n\pi)\} - \{\cos n\pi - 1\}\right] = \frac{2}{n\pi}[1-(-1)^n]$$

$$\text{i.e. } b_n = \begin{cases} \frac{4}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Now fourier series $f(x) = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \ldots\ldots\ldots$

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

Fourier Series for Arbitrary Interval

In many engineering problems, the period of the function required to be expanded is not 2π,
but some other interval, say: $2L$.

Euler formula for fourier series in the interval $c < x < c + 2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \, con \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where: a_0, a_n *and* b_n *are constant, known as fourier coefficient,*
which can be evaluated by Euler formula

$$a_0 = \frac{1}{L} \int_{c}^{c+2L} f(x)\,dx, \quad a_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos \frac{n\pi x}{L} dx, \; b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L} dx$$

Case.I : If $c = 0$ then fourier series in $0 < x < 2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{con} \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx \;,\; a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \& \quad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Case.II : If $c = -L$ then fourier series in $-L < x < L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{con} \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,\; a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \& \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Example : Obtain the fourier series for $f(x) = \begin{cases} \pi x , & 0 \le x \le 1 \\ \pi(2-x) , & 1 \le x \le 2 \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots = \frac{\pi^2}{8}$.

Solution : Here we can take L = 1 so that interval become (0,2L)

$$\text{Now } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{ con } \frac{n\pi x}{1} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1} \qquad \dots(1)$$

$$\text{where } a_0 = \int_0^2 f(x)\,dx \text{ , } a_n = \int_0^2 f(x) \cos \frac{n\pi x}{1} dx \quad \& \quad b_n = \int_0^2 f(x) \sin \frac{n\pi x}{1} dx$$

$$\text{now } a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \pi$$

$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{1}\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left(\pi x\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right)_0^1 + \left(\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right)_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1] = \begin{cases} 0 \text{ when n is even} \\ -\dfrac{4}{n^2\pi} \text{ when n is odd} \end{cases}$$

$$b_n = \int_0^2 f(x)\sin\frac{n\pi x}{1}\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$b_n = \int_0^2 f(x)\sin\frac{n\pi x}{1}\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left(\pi x\left(\frac{-\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right)_0^1 + \left(\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right)_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

$$\text{Hence } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\ldots\ldots\infty\right)$$

$$\text{putting} x = 2,\;\; 0 = \frac{\pi}{2}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \ldots\ldots\ldots\ldots\infty\right)$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}.$$

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

Theorem. The Fourier series corresponding to $f(x)$ may be integrated term by term from a to x, and the resulting series will converge uniformly to $\int_a^x f(x)\,dx$ provided that $f(x)$ is piecewise continuous in $-L \leqq x \leqq L$ and both a and x are in this interval.

PARSEVAL'S IDENTITY

If a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

Then

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent) we obtain

$$\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (1)$$

where we have used the results

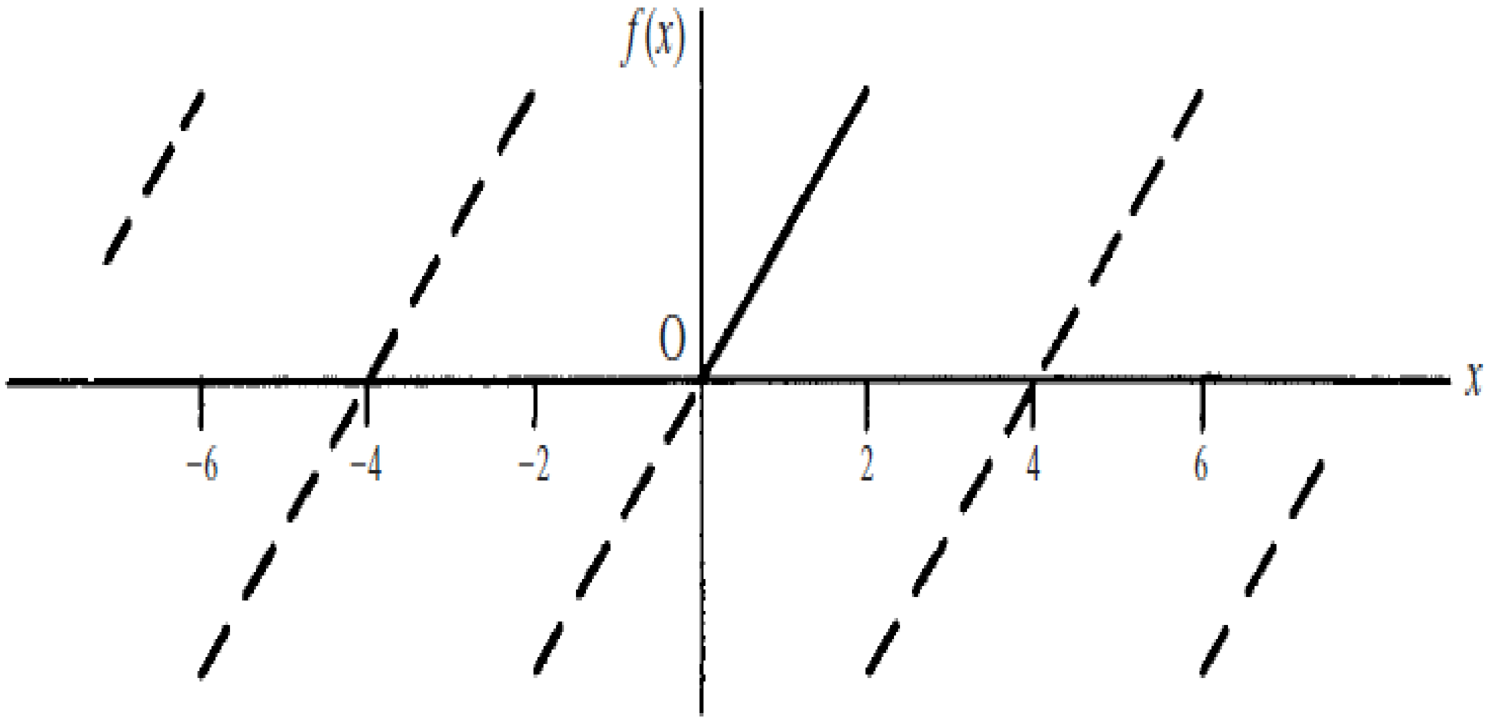
$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = Lb_n, \qquad \int_{-L}^{L} f(x)\,dx = La_0 \qquad (2)$$

obtained from the Fourier coefficients.

Example

Expand $f(x) = x, 0 < x < 2$, in a half range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4



Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

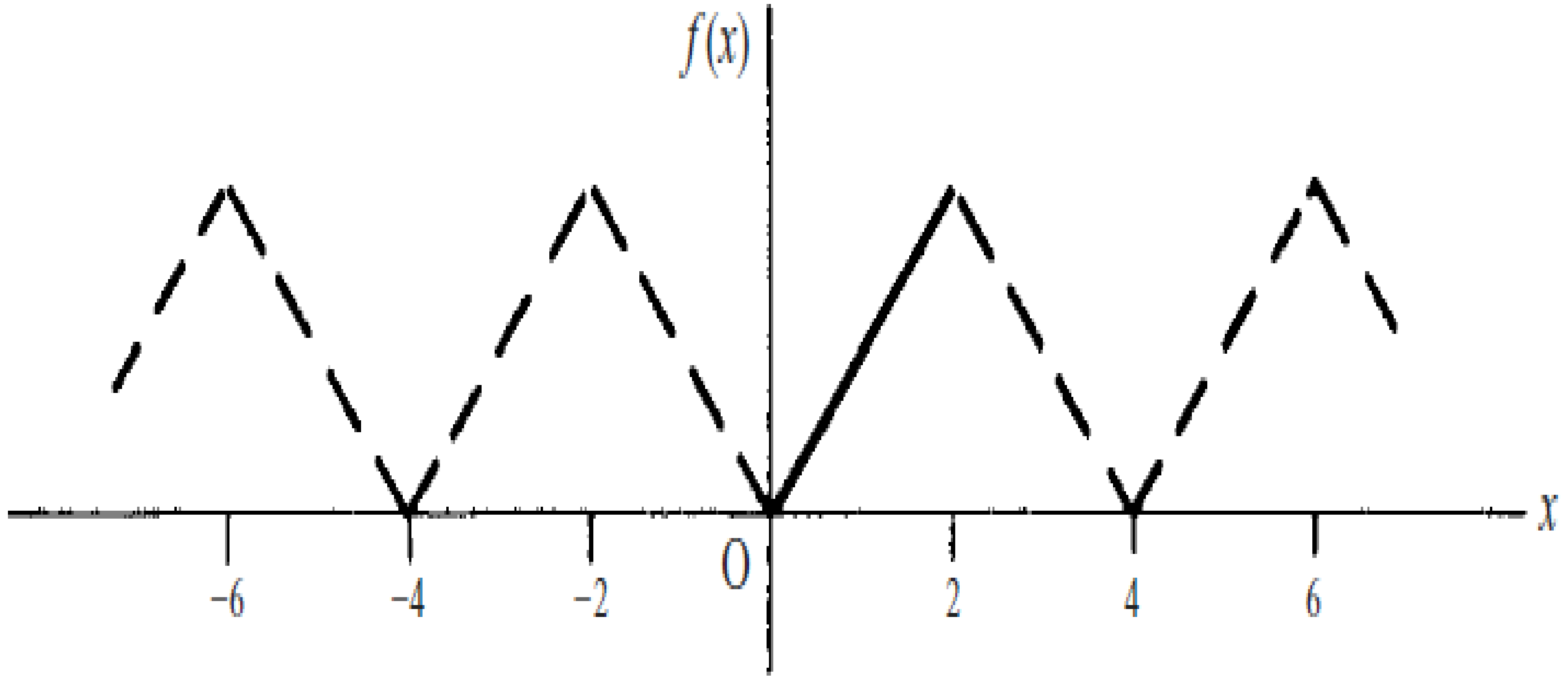
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then

$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(b) Extend the definition of $f(x)$ to that of the even function of period 4



Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{If } n \neq 0$$

If $n = 0$, $a_0 = \int_0^2 x\,dx = 2$.

Then $$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that the given function $f(x) = x$, $0 < x < 2$, is represented *equally well* by the two *different* series in (a) and (b).

Example

(a) Write Parseval's identity corresponding to the Fourier series of Problem (b).

(b) Determine from (a) the sum S of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{n^4}+\cdots$.

(a) Here $L=2$, $a_0=2$, $a_n=\frac{4}{n^2\pi^2}(\cos n\pi-1)$, $n\neq 0$, $b_n=0$.

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2\,dx=\frac{1}{2}\int_{-2}^{2}x^2\,dx=\frac{(2)^2}{2}+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi-1)^2$$

or $\frac{8}{3}=2+\frac{64}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$, i.e., $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}$.

$$(b)\quad S=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right)$$

$$=\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right)$$

$$=\frac{\pi^4}{96}+\frac{S}{16},\quad \text{from which } S=\frac{\pi^4}{90}$$