Module 4: Lecture 14 to 18
Reynolds Transport Theorem and Navier-Stoke's Equation

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Topics Covered:

1. Reynolds Transport Theorem (RTT).
   - Statement and Derivation of RTT.
   - Integral Form of Conservation of Mass Equation.
2. Conservation of Linear Momentum (Origin of Navier-Stoke's Equation).
   - Derivation of Navier's Equation.
   - Stoke's Hypothesis.
   - Derivation of Navier-Stoke's Equation.

1 Reynolds Transport Theorem (RTT)

This theorem is also known as the **general conservation law** because of the fact that *it provides general form for converting conservation laws from control mass to control volume.*

Consider a control system at time t with some identified mass as shown in the figure. Let the fixed mass occupies the space at time $t + \Delta t$ as shown in the figure.
Let N be the property *(extensive property)* that is being conserved and let n be the value of N per unit mass *(specific property)*. Therefore;

$$N_t = (N_I)_t + (N_{II})_t,$$
$$\text{and} \quad N_{t+\Delta t} = (N_{II})_{t+\Delta t} + (N_{III})_{t+\Delta t}$$

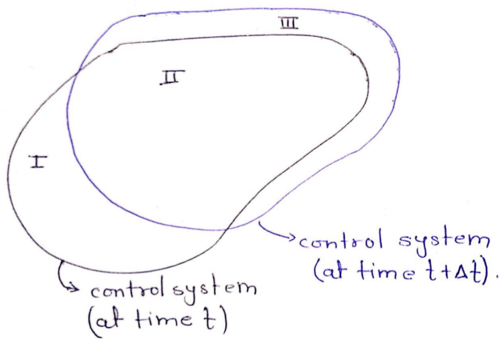

Figure 1: depicting the deformation of control system in Δt time

Therefore;

$$\left.\frac{dN}{dt}\right|_{system} = \lim_{\Delta \to 0} \frac{N_{t+\Delta t} - N_t}{\Delta t}$$

$$= \lim_{\Delta \to 0} \frac{(N_{II})_{t+\Delta t} - (N_{II})_t}{\Delta t} + \lim_{\Delta \to 0} \frac{(N_{III})_{t+\Delta t}}{\Delta t} - \lim_{\Delta \to 0} \frac{(N_I)_t}{\Delta t}$$

$$\implies \frac{dN}{dt}\Big|_{system} = \frac{\partial N}{\partial t}\Big|_{C.V.} + \text{Rate of outflow of } N - \text{Rate of inflow of } N \tag{1}$$

Note that in above equation, we are just writing $\frac{\partial}{\partial t}$ (partial derivative) of N, because it can also be function of different variables. Now;

$$\text{mass flow rate from the elementary area } dA = \rho(\vec{q}.\hat{\eta})dA$$
$$\text{rate of flow of quantity } N = n\rho(\vec{q}.\hat{\eta})dA$$
$$\therefore \textit{total rate of flow of } N = \int_{C.S.} n\rho(\vec{q}.\hat{\eta})dA.$$

Here the total rate of flow includes both rate of outflow & rate of in flow. Also note that the abbreviated terms $C.V.$ denotes the **Control Volume** whereas $C.S.$ denotes the **Control Surface**. Now, using the value of total rate of flow of N in equation (1), we can write;

$$\frac{dN}{dt}\Big|_{system} = \frac{\partial N}{\partial t}\Big|_{C.V.} + \int_{C.S.} n\rho(\vec{q}.\hat{\eta})dA$$

$$\text{As mass for elementary volume } dV = \rho dV$$
$$\implies N = \int_{C.V.} \rho n dV.$$

Therefore we get;

$$\frac{dN}{dt}\Big|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho n dV + \int_{C.S.} n\rho(\vec{q}.\hat{\eta})dA$$

If the control volume is also moving and let $\vec{q_r}$ be the relative velocity of the fluid within the control volume then;

$$\frac{dN}{dt}\Big|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho n dV + \int_{C.S.} \rho n(\vec{q_r}.\hat{\eta})dA \tag{2}$$

This equation states the **Reynolds transport theorem** or **RTT**.

1.1 Integral Form of Conservation of Mass Equation

By the Reynolds transport theorem *i.e.* equation (2), taking $N = m$ (mass) and therefore $n = 1$ ($\because n = N$ per unit mass) we get;

$$\frac{dm}{dt}\Big|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho dV + \int_{C.S.} \rho(\vec{q_r}.\hat{\eta})dA$$

$$\text{As we have considered a control system and therefore we have } \frac{dm}{dt}\Big|_{system} = 0$$

$$\implies \frac{\partial}{\partial t}\int_{C.V.} \rho dV + \int_{C.S.} \rho(\vec{q_r}.\hat{\eta})dA = 0.$$

Let the control volume be non-deformable (*i.e.* no change *w.r.t.* time) and control volume is also assumed as stationary (*i.e.* $\vec{q_r} = \vec{q}$);

$$\therefore \int_{C.V.} \frac{\partial \rho}{\partial t} dV + \int_{C.S.} \rho\vec{q}.\hat{\eta}dA = 0$$

Now applying divergence theorem *i.e.* $\int_S (\vec{F}.\hat{n})dA = \int_V (\nabla \vec{F})dV$ we get;

$$\int_{C.V.} \frac{\partial \rho}{\partial t} dV + \int_{C.V.} \nabla(\rho \vec{q}) dV = 0$$
$$\implies \int_{C.V.} \left[\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{q}) \right] dV = 0. \tag{3}$$

Since the elementary volume dV is arbitrarily chosen, so if integral over dV is zero then integrand is zero. Therefore from equation (3) we get;

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{q}) = 0 \tag{4}$$

The above equation is the conservation of mass equation or the equation of continuity.

2 Conservation of Linear Momentum (Origin of Navier-Stoke's Equation)

Let us assume that control volume is non-deformable and independent of t. We have $N = m\vec{q} = linear\ momentum$ and $n = \vec{q}$.
Therefore by the Reynolds transport theorem, we have;

$$\left. \frac{d(m\vec{q})}{dt} \right|_{system} = \frac{\partial}{\partial t} \int_{C.V.} \rho \vec{q} dV + \int_{C.S.} \rho \vec{q} (\vec{q}.\hat{\eta}) dA \tag{5}$$

Further we have;

$$\left. \frac{d(m\vec{q})}{dt} \right|_{system} = m. \left. \frac{d\vec{q}}{dt} \right|_{system} = \sum \vec{F} \Big|_{system}$$

In the limiting case, when $\Delta t \to 0$, we know that system approaches the control volume;

$$\therefore \sum \vec{F} \Big|_{C.V.} = \frac{\partial}{\partial t} \int_{C.V.} \rho \vec{q} dV + \int_{C.S.} \rho \vec{q} (\vec{q}.\hat{\eta}) dA \tag{6}$$

Writing $x - component$ of the equation (6);

$$\sum F_{i,C.V.} = \frac{\partial}{\partial t} \int_{C.V.} \rho u_i dV + \int_{C.S.} \rho u_i (\vec{q}.\hat{\eta}) dA \tag{7}$$

We have, resultant force = surface force + body force *i.e.*;

$$\sum F_{i,C.V.} = \int_{C.S.} T_i^{\eta} dA + \int_{C.V.} \rho b_i dV \tag{8}$$

Further we know that,

$$T_i^{\eta} = \tau_{i1}\eta_1 + \tau_{i2}\eta_2 + \tau_{i3}\eta_3$$
$$\implies T_i^{\eta} = (\tau_{i1}\hat{i} + \tau_{i2}\hat{j} + \tau_{i3}\hat{k}).(\eta_1\hat{i} + \eta_2\hat{j} + \eta_3\hat{k})$$
$$\implies T_i^{\eta} = \vec{\tau_i}.\hat{\eta}$$

$$\therefore \text{ equation (8) can be written as,} \qquad \sum F_{i,C.V.} = \int_{C.S.} (\vec{\tau_i}.\hat{\eta})dA + \int_{C.V.} \rho b_i dV$$

$$\text{Using divergence theorem,} \qquad \sum F_{i,C.V.} = \int_{C.V.} (\nabla\vec{\tau_i})dV + \int_{C.V.} \rho b_i dV \tag{9}$$

Now using equation (9) in equation (7) we get;

$$\int_{C.V.} (\nabla\vec{\tau_i})dV + \int_{C.V.} \rho b_i dV = \int_{C.V.} \rho u_i dV + \int_{C.S.} \rho u_i(\vec{q}.\hat{\eta})dA$$

$$\implies \int_{C.V.} (\nabla\vec{\tau_i})dV + \int_{C.V.} \rho b_i dV = \int_{C.V.} \rho u_i dV + \int_{C.V.} \nabla(\rho u_i \vec{q})dV \quad \text{[using divergence theorem]}$$

Further solving the above equation;

$$\implies \int_{C.V.} \left[\frac{\partial}{\partial t}(\rho u_i) + \nabla(\rho u_i \vec{q}) - (\nabla\tau_i) - \rho b_i\right] dV = 0$$

$$\implies \frac{\partial}{\partial t}(\rho u_i) + \nabla(\rho u_i \vec{q}) - (\nabla\tau_i) - \rho b_i = 0$$

In index notation;

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}\tau_{ij} + \rho b_i \tag{10}$$

We can represent,

$$\tau_{ij} = \tau_{ij}^{hydrostatic} + \tau_{ij}^{deviatoric}. \tag{11}$$

Now, ***for further analyzing*** $\tau_{ij}^{deviatoric}$***:*** consider;

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}_{e_{ij}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]}_{w_{ij}}.$$

Clearly, e_{ij} is related with the shearing stain rate, $e_{ij} = e_{ji}$ (*i.e.* it's a symmetric tensor). Also, w_{ij} is related with the rotation and $w_{ij} = -w_{ji}$ (*i.e.* it's a skew-symmetric tensor).

Since $\tau_{ij}^{deviatoric}$ is responsible for the shear deformation of the fluid element. Therefore, we have the constitutive relation for the particular fluid;

$$\tau_{ij}^{deviatoric} = f(e_{ij}).$$

Note: For mapping vector to vector, we needed 2^{nd} order tensor & for mapping 2^{nd} order tensor to 2^{nd} order tensor, we need 4^{th} order tensor.

Assuming, the fluid be Newtonian fluid we have;

$$\tau_{ij}^{deviatoric} = c_{ijkl}e_{kl}. \tag{12}$$

Further, let us assume that the fluid is homogeneous (position independent) and isotropic (direction independent).

$$
\begin{aligned}
\implies c_{ijkl}A_iB_jC_kD_l &= s \ \ (scalar) \\
&= \alpha(\vec{A}.\vec{B})(\vec{C}.\vec{D}) + \beta(\vec{A}.\vec{C})(\vec{B}.\vec{D}) + \gamma(\vec{A}.\vec{D})(\vec{B}.\vec{C}) \\
&= \alpha(A_i.B_i)(C_k.D_k) + \beta(A_i.C_i)(B_j.D_j) + \gamma(A_i.D_i)(B_j.C_j) \\
&= \alpha(A_i\delta_{ij}B_j)(C_k\delta_{kl}D_l) + \beta(A_i\delta_{ik}C_k)(B_j\delta_{jl}D_l) + \gamma(A_i\delta_{il}D_l)(B_j\delta_{jk}C_k).
\end{aligned}
$$

Comparing;

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}.$$

Here α, β and γ are position independent constants.

We also know that;

$$
\begin{aligned}
\tau_{ij}^{deviatoric} &= \tau_{ji}^{deviatoric} \\
\implies c_{ijkl} &= c_{jikl} && \text{[using (12)]} \\
\implies \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} &= \alpha\delta_{ji}\delta_{kl} + \beta\delta_{jk}\delta_{il} + \gamma\delta_{jl}\delta_{ik} \\
\implies \beta &= \gamma.
\end{aligned}
$$

Therefore;

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk}$$

Now,

$$
\begin{aligned}
\tau_{ij}^{deviatoric} &= c_{ijkl}.e_{kl} \\
&= \big[\alpha\delta_{ij}\underbrace{\delta_{kl}}_{k=l} + \beta\underbrace{\delta_{ik}\delta_{jl}}_{k=i,l=j} + \beta\underbrace{\delta_{il}\delta_{jk}}_{l=i,k=j}\big].e_{kl} \\
&= \alpha\delta_{ij}e_{kk} + \beta e_{ij} + \beta e_{ji}. \\
\therefore \tau_{ij}^{deviatoric} &= \alpha\delta_{ij}e_{kk} + 2\beta e_{ij} \qquad (13)
\end{aligned}
$$

Note: *Let us understand what is* e_{kk}*;*

$$
\begin{aligned}
e_{ij} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \\
\implies e_{kk} &= \frac{\partial u_k}{\partial x_k} \\
&= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\
\therefore e_{kk} &= \nabla.\vec{q}.
\end{aligned}
$$

Again writing, equation (13) as;

$$\tau_{ij}^{deviatoric} = \lambda\delta_{ij}e_{kk} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

where, $\mu = \beta =$ coefficient of viscosity & $\lambda = \alpha =$ second coefficient of viscosity.

and

$$\tau_{ij}^{hydrostatic} = -p\delta_{ij}.$$

here, negative sign is for normal component of force/pressure which is opposite to the normal direction.

Therefore, from equation (11), we have a constitutive relationship for a homogeneous, isotropic Newtonian fluid;

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}e_{kk} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Now, assuming that the partial derivatives are continuous;

$$\begin{aligned}
\frac{\partial}{\partial x_j}\tau_{ij} &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\lambda e_{kk}) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \\
&= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\lambda e_{kk}) + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) \\
&= -\frac{\partial p}{\partial x_i} + \lambda\frac{\partial}{\partial x_i}(e_{kk}) + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\
\implies \frac{\partial}{\partial x_j}\tau_{ij} &= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right). \qquad (14)
\end{aligned}$$

Using equation (14) in the equation (10), we get;

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \rho b_i. \qquad (15)$$

This (above) equation is ***conservative form of the Navier's equation*** for a homogeneous, isotropic Newtonian fluid.

Let us do some more gymnastics (simplifications) for getting the ***non-conservative form of the Navier's equation***: simplifying the left side of the equation (15), we get;

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) &= \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial}{\partial x_j}(\rho u_j) \\
&= \rho\frac{\partial u_i}{\partial t} + u_i\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}\right] + \rho u_j\frac{\partial u_i}{\partial x_j} \\
&= \rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] \\
\implies \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) &= \rho\frac{Du_i}{Dt}
\end{aligned}$$

Therefore, equation (1.16) becomes;

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \rho b_i \qquad (16)$$

This (above) equation is ***non-conservative form of the Navier's equation*** for a homogeneous, isotropic Newtonian fluid.

2.1 Stoke's Hypothesis

After doing some more gymnastics with the stress components, we will apply the ***Stoke's assumptions (for Stoke's-ian fluids)***;

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}e_{kk} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Taking normal components:

$$\tau_{11} = -p + \lambda e_{kk} + 2\mu\frac{\partial u_1}{\partial x_1}$$
$$\tau_{22} = -p + \lambda e_{kk} + 2\mu\frac{\partial u_2}{\partial x_2}$$
$$\tau_{33} = -p + \lambda e_{kk} + 2\mu\frac{\partial u_3}{\partial x_3}.$$

Adding all three stress components, we get;

$$\tau_{11} + \tau_{22} + \tau_{33} = -3p + 3\lambda e_{kk} + 2\mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$
$$\implies \tau_{11} + \tau_{22} + \tau_{33} = -3p + 3\lambda e_{kk} + 2\mu e_{kk}$$
$$\implies \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} = -p + \left(\lambda + \frac{2}{3}\mu\right)e_{kk}.$$

Here, p is the *thermodynamic pressure* and the term $-\frac{\tau_{11}+\tau_{22}+\tau_{33}}{3} = p_m$ is the *mechanical pressure* and therefore, we have;

$$-p_m = -p + \left(\lambda + \frac{2}{3}\mu\right)e_{kk}.$$

From this, we can conclude that: the *mechanical pressure* p_m and the *thermodynamic pressure* p are equal if;

1. $e_{kk} = 0$ *(flow is incompressible).*
2. $\lambda = -\frac{2}{3}\mu$ *(**Stoke's hypothesis**).*

Note: *Stoke's-ian fluid* is the fluid that satisfies *Stoke's hypothesis* (mentioned above).

Therefore, for a *Stoke's-ian fluid* we have $\lambda = -\frac{2}{3}\mu$ and thus equation (16) becomes;

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \rho b_i \tag{17}$$

If flow is incompressible *i.e.* $e_{kk} = 0$, and therefore;

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \rho b_i. \tag{18}$$

This (above) equation is the ***simplest form of the Navier-Stoke's equation***.

Writing the $x - component$ (similarly we can write other components) of this equation;

$$\rho \frac{Du_1}{Dt} = -\frac{\partial p}{\partial x_1} + \mu \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\frac{\partial u_1}{\partial x_j}\right) + \rho b_1$$

$$Or \qquad \rho \frac{Du_1}{Dt} = -\frac{\partial p}{\partial x_1} + \mu \left[\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right] + \rho b_1.$$

Vector form of the Navier-Stoke's equation can be written as;

$$\rho \frac{D\vec{q}}{Dt} = -\nabla p + \mu \nabla^2 \vec{q} + \rho \vec{b}. \tag{19}$$

Thus we are done with the Navier-Stoke's equation for a viscous, incompressible, homogeneous, isotropic Newtonian fluid flow. Further for our satisfaction, we can check the *dimensional homogeneity* of this equation.

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