# *Module 3: Lecture 10 to 13*
# ***Continuity Equation, Angular Strain Rate and Euler's Equation of Motion***

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**Topics Covered:**

1. Continuity Equation (Conservation of mass).
   - Control System Approach.
   - Control Volume Approach.
2. Angular Strain Rate (Angular Deformation of Fluid Particle).
   - Different Cases: When Flow is 2D, Incompressible and Irrotational.
3. Dynamics of Inviscid Fluid.
   - Euler's Equation of Motion.
   - Bernoulli's Equation of Motion.

## 1 Continuity Equation (Conservation of Mass)

Here in this section, we will do some gymnastics for driving an equation which enforces conservation of mass *i.e.* the *continuity equation*.

⋆ ***Continuity Equation*** ⋆

In fluid dynamics, the ***continuity equation*** states that ***the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system***.

⋆ ***Differential Form of the Continuity Equation*** ⋆

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{q}) = 0.$$

here, $\rho$ is the fluid density, $t$ is the time and $\vec{q}$ is the flow velocity vector field.

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***Note:*** *The* ***Navier-Stoke's equation*** *form a* ***vector continuity equation*** *describing the* ***conservation of linear momentum*** *(which wiil be discussed in the* ***module 4****).*

This is not strictly a description of material behavior, but the resulting equation is often used as an identity to algebraically manipulate constitutive models describing material behavior. So ***it is worth reviewing***. It is also central to the analysis of fluid flow because classical fluids analyses cannot be Lagrangian since the positions of all the fluid particles at $t = 0$ are unknown.

## 1.1 Control System Approach

Let us consider a *control system* with mass $m$, Then;

$$m = \rho V$$
$$\implies \log m = \log \rho + \log V$$

Taking *time derivative (actually the* ***material derivative***[1] *)* to the both sides of above equation *(keeping in mind that the mass is treated as constant due to control system)*, we get;

$$\frac{1}{m}\cancelto{0}{\frac{Dm}{Dt}} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{V}\frac{DV}{Dt}$$

$$\implies \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{V}\frac{DV}{Dt} = 0 \tag{1}$$

***Note:*** The second term in equation (1) is nothing but *the rate of change of volume per unit time, per unit volume i.e.* ***volumetric strain rate***. And that's the reason why we replaced it with the term $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ in next step *(for more details please refer to the* ***module 2****)*.

$$\implies \frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right] + \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

$$\implies \left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right] + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

$$\implies \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\implies \frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{q}) = 0 \tag{2}$$

This is the required ***differential form of the continuity equation***. The *time derivative* in equation (2) can be understood as the accumulation (or loss) of mass in the system, while the *divergence term* represents the difference in flow in versus flow out. In this context, this equation is also one of the Euler's equations (fluid dynamics).

***If the flow is incompressible:*** Flow incompressible means the volumetric strain rate is zero, *i.e.* $\frac{1}{V}\frac{DV}{Dt} = 0$ in equation (1) $\implies \frac{1}{\rho}\frac{D\rho}{Dt} = 0 \implies \frac{D\rho}{Dt} = 0$.

[1] The ***material derivative*** is defined for any tensor field $Q$ that is macroscopic, with the sense that it depends only on position and time coordinates, $Q = Q(x, t)$ :

$$\frac{DQ}{Dt} \equiv \frac{\partial Q}{\partial t} + \vec{q}.\nabla Q$$

where $\nabla Q$ is the *covariant derivative* of the tensor, and $\vec{q}$ is the flow velocity.

## 1.2 Control Volume Approach

The equation is developed by adding up the rate at which mass is flowing in and out of a control volume, and setting the net in-flow equal to the rate of change of mass within it. This is demonstrated in the figure on right side and mathematically we can write it as;

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t}(m_{CV}). \tag{3}$$

where $\dot{m}_{in}$ is the mass flow rate that flows in, $\dot{m}_{out}$ is the mass flow rate that flows out and $m_{CV}$ is the rate of change of mass in control volume.

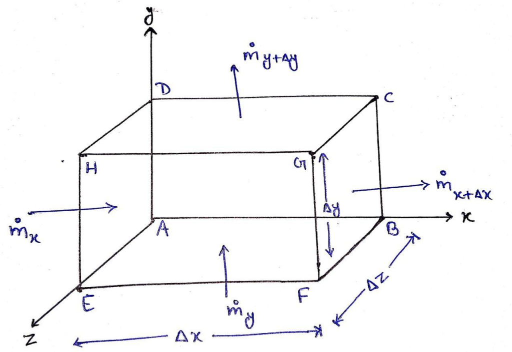


Figure 1: depicting the mass flow rates in control volume.

Now, let us do some gymnastics with equation (3);

$$[\dot{m}_x - \dot{m}_{x+\Delta x}] + [\dot{m}_y - \dot{m}_{y+\Delta y}] + [\dot{m}_z - \dot{m}_{z+\Delta z}] = \frac{\partial}{\partial t}(m_{CV}). \tag{4}$$

Now, using the Taylor's series expansion;

$$\begin{aligned} \dot{m}_x - \dot{m}_{x+\Delta x} &= \dot{m}_x - \left(\dot{m}_x + \frac{\partial \dot{m}_x}{\partial x}\Delta x + ...\right) \\ &= -\frac{\partial \dot{m}_x}{\partial x}\Delta x - ... \end{aligned}$$

Similarly, we can write others;

$$\begin{aligned} \dot{m}_y - \dot{m}_{y+\Delta y} &= -\frac{\partial \dot{m}_y}{\partial y}\Delta y - ... \\ \dot{m}_z - \dot{m}_{z+\Delta z} &= -\frac{\partial \dot{m}_z}{\partial z}\Delta z - ... \end{aligned}$$

Putting all these (ignoring the higher order terms) in equation (4), we get;

$$-\left[\frac{\partial \dot{m}_x}{\partial x}\Delta x + \frac{\partial \dot{m}_y}{\partial y}\Delta y + \frac{\partial \dot{m}_z}{\partial z}\Delta z\right] = \frac{\partial}{\partial t}(m_{CV}). \tag{5}$$

also we have;

$$\begin{aligned} \dot{m}_x &= \rho \Delta y.\Delta z.u \\ \dot{m}_y &= \rho \Delta z.\Delta x.v \\ \dot{m}_z &= \rho \Delta x.\Delta y.w \\ m_{CV} &= \rho \Delta x.\Delta y.\Delta z \end{aligned}$$

Therefore from equation (5), we get;

$$\begin{aligned} -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right](\Delta x \Delta y \Delta z) &= \frac{\partial}{\partial t}\rho(\Delta x \Delta y \Delta z) \\ \implies -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] &= \frac{\partial \rho}{\partial t} \end{aligned}$$

$$\implies \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$
$$\implies \frac{\partial \rho}{\partial t} + \nabla(\rho\vec{q}) = 0.$$

And thus, again we get the ***differential form of the continuity equation***.

***If the flow is incompressible:*** That means, if density is not variable $x, y, z, t \implies \frac{\partial \rho}{\partial t} = 0$ and that further $\implies \nabla.\vec{q} = 0$.

***Example 1.1:*** *Check whether the flow is possible or not for the given velocity field $\vec{q} = x\hat{i} - y\hat{j}$.*
***Explanation:*** As $\nabla.\vec{q} = 0$ for the given velocity field so the *flow is possible or $\vec{q}$ is the possible velocity field.* Further, note that if the equation of continuity is not satisfied then we say *flow is not possible* (for an example take $\vec{q} = x\hat{i} - y\hat{j} + z\hat{k}$).

## 2 Angular Strain Rate (Angular Deformation of Fluid Particle)

Consider a fluid element $ABCD$ as shown in the figure with dimensions $\Delta x$ and $\Delta y$. Let after $\Delta t$ time the fluid element deforms and occupies the shape of PQRS.

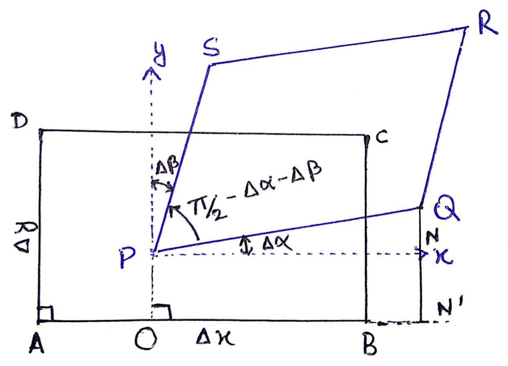

Figure 2: depicting the angular deformation of fluid particle.

We have $OP = v.\Delta t$ and $QN' = v(x + \Delta x).\Delta t$. By using Taylor's series expansion we have;

$$QN' = \left[v + \frac{\partial v}{\partial x}.\Delta x + ...\right]\Delta t$$

now,

$$QN = QN' - NN'$$

or

$$QN = QN' - OP$$

Then by using values of $QN'$ and $OP$ we will get,

$$QN = \frac{\partial v}{\partial x}\Delta x \Delta t + ...$$

also we have

$$PN = \Delta x + ...$$

$\therefore$ in $\triangle PNQ$ we have,

$$\tan(\Delta\alpha) = \frac{QN}{PN}$$
$$\implies \tan(\Delta\alpha) = \frac{\frac{\partial v}{\partial x}\Delta x\Delta t + ...}{\Delta x + ...} \tag{6}$$

If $\Delta t \to 0$ then $\Delta\alpha \to 0$, in that case $\tan(\Delta\alpha) \approx \Delta\alpha$ and therefore from equation (6) we get;

$$\lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} = \frac{\partial v}{\partial x}$$
$$\implies \dot{\alpha} = \frac{\partial v}{\partial x}$$

similarly, doing whole process along $y - coordinate$ we can get,

$$\dot{\beta} = \frac{\partial u}{\partial y}$$

where $\dot{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}$ and $\dot{\beta} = \lim_{\Delta t \to 0} \frac{\Delta \beta}{\Delta t}$. $u$ and $v$ velocity components in $x$ and $y$ directions respectively.

***Angular Strain Rate:*** The angular strain rate is define as rate of change of angle between the line elements which were initially perpendicular to each other.

In our case, *angular strain rate is basically the rate of change of $\angle SPQ$ with respect to time.*

$$\therefore \text{Change in angle} = \frac{\pi}{2} - \left(\frac{\pi}{2} - \Delta\alpha - \Delta\beta\right) = \Delta\alpha + \Delta\beta.$$

$$\begin{aligned}\therefore \text{Rate of shear strain or rate of angular strain} &= \frac{\Delta\alpha + \Delta\beta}{\Delta t} \\ &= \dot{\alpha} + \dot{\beta} \\ &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\end{aligned}$$

***Angular Velocity:*** We define the angular velocity of the fluid element as;

$$\begin{aligned}\omega_z &= \frac{1}{2}(\dot{\alpha} - \dot{\beta}) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).\end{aligned}$$

***Vorticity:*** Vorticity is a *pseudovector (a vector under a proper angular rotation)*, which is denoted by $\vec{\boldsymbol{\Omega}}$ and defined by;

$$\begin{aligned}&\vec{\boldsymbol{\Omega}} = curl\ \vec{q} = \nabla \times \vec{q} \\ &\implies \vec{\boldsymbol{\Omega}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &\implies \vec{\boldsymbol{\Omega}} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &\implies \vec{\boldsymbol{\Omega}} = \hat{i}(2\omega_x) + \hat{j}(2\omega_y) + \hat{k}(2\omega_z) \\ &\implies \vec{\boldsymbol{\Omega}} = 2\vec{\omega} \\ &\therefore vorticity = 2 \times angular\ velocity\end{aligned}$$

where $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ is the angular velocity vector.

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***Note:*** Vorticity vector is just effect of angular velocity. Moreover, flow vorticity$= 0$ if and only if flow is irrotational *or* irrotational flow $\iff curl\vec{q} = \vec{0}$.

## 2.1 Flow Analysis

- **Let the flow is 2-D and incompressible:** The equation of continuity is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Let us define $\psi = \psi(x, y)$ such that;

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

  Then the function $\psi$ is called ***stream function***. Consider $\psi = constant$ (represents a family of curve), then;

$$\begin{aligned} &\implies \partial\psi = 0 \\ &\implies \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 \\ &\implies -vdx + udy = 0 \\ &\implies \frac{dx}{u} = \frac{dy}{v} \end{aligned}$$

  Therefore $\psi = constant$, represents streamlines of the flow.

- **Let the flow is irrotational:** Let $\phi(x, y, z)$ be a scalar function;

$$\therefore curl(\nabla\phi) = \vec{0}.$$

  Comparing both of the above equations, we can say that $\exists$ a scalar function $\phi(x, y, z)$ such that;

$$\begin{aligned} \vec{q} &= \nabla\phi \\ \implies (u, v, w) &= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \\ \implies u = \frac{\partial \phi}{\partial x}, \ v &= \frac{\partial \phi}{\partial y}, \ w = \frac{\partial \phi}{\partial z}. \end{aligned}$$

  ***Note:*** The scalar function $\phi(x, y, z)$ is called ***velocity potential***. If the fluid flow is irrotational and inviscid then flow will always remain irrotational and such type of flow is called ***potential flow***.

- **Let the flow be 2D, incompressible and irrotational:** That is stream function $\psi = \psi(x, y)$ exist and $\psi = constant$.

$$\begin{aligned} &\implies d\psi = 0 \\ &\implies \frac{dx}{u} = \frac{dy}{v} \\ &\implies \frac{dy}{dx} = \frac{v}{u} \\ &\implies slope|_{\psi=c} = \frac{v}{u}. \end{aligned}$$

Also $\phi = constant \implies d\phi = 0.$

$$\implies \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$
$$\implies udx + vdy = 0$$
$$\implies \frac{dy}{dx} = -\frac{u}{v}$$
$$\implies slope|_{\phi=k} = -\frac{u}{v}.$$

At the point of intersection of $\psi = c$ and $\phi = k$, we have;

$$(slope|_{\psi=c}).(slope|_{\phi=k}) = -1.$$

$\implies$ $\psi = c$ and $\phi = k$ are orthogonal to each other.

***Note:*** For 2D, incompressible flow, $u = \frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$. If the flow is also irrotational then $\omega_z = 0$, that is we have;

$$\frac{1}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) = 0$$
$$\implies \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$\implies \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

$\implies$ Stream function satisfies the Laplace equation.

For 2D irrotational flow, $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$. If the flow is also incompressible, then we have;

$$\implies \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\implies \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

$\implies$ $\phi$ also satisfies the Laplace equation.

⋆***Exercise 2.1***⋆

***Question:*** *Test whether the motion specified by* $\vec{q} = \frac{k^2(x\hat{j} - y\hat{i})}{x^2+y^2}$; $(k = constant)$, *is a possible motion for an incompressible fluid. If so, determine the equation of streamlines. Also test whether the motion is of potential kind and if so, obtain the velocity potential?*

***Hint:*** For checking the possibility of the flow we need to just check that $\nabla.\vec{q} = 0$ is satisfied or not. Then for streamlines, first find the stream function $\psi$ from $u = \frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$ and then find the equation of streamlines from $\psi = \psi(x, y)$. Finally for checking whether the flow is of potential kind or not, we need to just verify $\nabla \times \vec{q} = 0$ and then one can find velocity potential $\phi$ from $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$.

# 3 Dynamics of Inviscid Fluid

## 3.1 Euler's Equation of Motion

Consider a fluid element as shown in the figure with dimensions $\Delta x$, $\Delta y$ and $\Delta z$. According to the *Newton's second law of motion*;

$$\sum F_x = m.a_x \tag{7}$$

where, $m$ = mass = $\rho\Delta x\Delta y\Delta z$ and $a_x$ is the material acceleration.

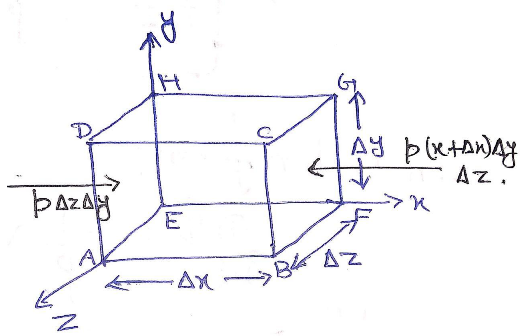


Figure 3: depicting the considered fluid element for Euler's equation of motion.

$$\begin{aligned}
\text{Force on the face} AEHD &= p(x)\Delta y\Delta z \\
\text{Force on the face} BFGC &= p(x+\Delta x)\Delta y\Delta z \\
&= \left[p(x) + \frac{\partial p}{\partial x}\Delta x + ...\right]\Delta y\Delta z.
\end{aligned}$$

Sum of surface forces on the fluid element is;

$$\begin{aligned}
&= p(x)\Delta y\Delta z - \left[p(x) + \frac{\partial p}{\partial x}\Delta x + ...\right]\Delta y\Delta z \\
&= -\left(\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z + ...\right).
\end{aligned}$$

Let $b_x$ be the body forcce per unit mass, then we have;

$$\text{Total body force} = \rho b_x\Delta x\Delta y\Delta z.$$

So *Total force* = *surface force* + *body force*, therefore from equation (7) we have;

$$\text{Surface force} + \text{body force} = m.a_x$$

$$\implies -\frac{\partial p}{\partial x}(\Delta x\Delta y\Delta z) + \rho b_x(\Delta x\Delta y\Delta z) = \rho(\Delta x\Delta y\Delta z)\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

$$\begin{aligned}
&\implies -\frac{\partial p}{\partial x} + \rho b_x = \rho\frac{Du}{Dt} \\
&\implies \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + b_x.
\end{aligned} \tag{8}$$

The above equation is the required ***Euler's equation of motion*** in $x-$*component*.

In ***vector form***;

$$\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{b}$$

*Or*

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \vec{b}. \tag{9}$$

This (above) equation is the ***Euler's equation of motion in vector form***.

***Example 3.1:*** *Consider the flow field given by* $\vec{q} = x\hat{i} - y\hat{j}$ and discuss the fluid motion.
***Explanation:*** Let us do gymnastics with the given flow field and discuss it point wise;

- Here $u = x$ and $v = -y$; so ***flow is clearly 2D***.
- Rate of shear strain $= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \implies$ *there is no shear deformation* $\implies$ the ***flow is inviscid***.
- As $\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0 \implies$ ***flow is irrotational*** $\implies$ ***flow is of potential kind***.
- As $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies$ ***flow is incompressible*** *or* ***volumetric strain rate is zero***.
- ***Streamlines:*** $\frac{dx}{u} = \frac{dy}{v} \implies \frac{dx}{x} = -\frac{dy}{y} \implies \frac{dx}{x} + \frac{dy}{y} = 0 \implies \log x + \log y = \log c \implies xy = c$; thus the ***streamlines are of rectangular hyperbola shape***.
- ***To find the pressure between any two points:*** using the vector form of Euler's equation of motions;

$$\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{b}$$

Assuming the body force is along $z-direction$ (gravity) *i.e.* $b_x = 0$ and $b_y = 0$. Writing the above equation component wise, we have;

$$\begin{aligned} &\textit{x-component:} && -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0 = [0 + x.1 + (-y).0 + 0] \implies -\frac{\partial p}{\partial x} = \rho x \\ &\textit{y-component:} && -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0 = [0 + x.0 + (-y)(-1) + 0] \implies -\frac{\partial p}{\partial y} = \rho y \\ &\textit{z-component:} && -\frac{1}{\rho}\frac{\partial p}{\partial z} - g = [0] \implies -\frac{\partial p}{\partial z} = \rho g \end{aligned}$$

further solving the *x-component*, we have;

$$-p = \frac{\rho x^2}{2} + f_1(y, z)$$

Further removing this arbitrary function $f_1(y, z)$ using other components of the equation, we can get the pressure $p$ as;

$$-p = \frac{\rho x^2}{2} + \frac{\rho y^2}{2} + \rho g z + c.$$

## 3.2 Euler's Equation along a Streamline (Bernoulli's Equation of Motion)

We have already derived the *Euler's equation of motion for Inviscid flow.* Here we will use the vector form of that equation (which is given by equation (9) in last subsection), writing the $x, y, z-$ *components* of that equation;

$$\frac{\partial p}{\partial x} = \rho\left[-\frac{\partial u}{\partial t} - (\vec{q}.\nabla)u + b_x\right]$$

$$\frac{\partial p}{\partial y} = \rho\left[-\frac{\partial v}{\partial t} - (\vec{q}.\nabla)v + b_y\right]$$

$$\frac{\partial p}{\partial z} = \rho\left[-\frac{\partial w}{\partial t} - (\vec{q}.\nabla)w + b_z\right]$$

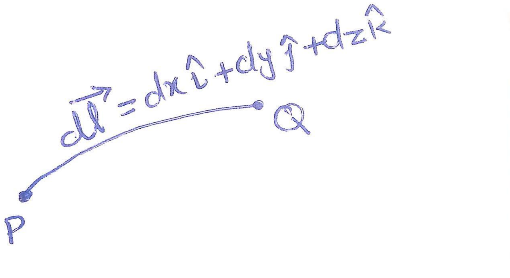


Figure 4: depicting the considered streamline for Euler's equation.

We also have, total differential of pressure;

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

$$\implies dp = \rho\left[-\frac{\partial u}{\partial t} - (\vec{q}.\nabla)u + b_x\right]dx + \rho\left[-\frac{\partial v}{\partial t} - (\vec{q}.\nabla)v + b_y\right]dy + \rho\left[-\frac{\partial w}{\partial t} - (\vec{q}.\nabla)w + b_z\right]dz$$

$$\implies dp = -\rho\left[\frac{\partial u}{\partial t}dx + \frac{\partial v}{\partial t}dy + \frac{\partial w}{\partial t}dz\right] - \rho\left[(\vec{q}.\nabla)udx + (\vec{q}.\nabla)vdy + (\vec{q}.\nabla)wdz\right]$$

$$+ \rho[\cancelto{0}{b_x}\; dx + \cancelto{0}{b_y}\; dy + \cancelto{-g}{b_z}\; dz]$$

$$\implies dp = -\rho\frac{\partial \vec{q}}{\partial t}.d\vec{l} - \rho\{(\vec{q}.\nabla)\vec{q}\}.d\vec{l} - \rho g dz$$

$$\implies dp = -\rho\frac{\partial \vec{q}}{\partial t}.d\vec{l} - \rho\left[\frac{1}{2}\nabla(\vec{q}.\vec{q}) - \vec{q}\times(\nabla\times\vec{q})\right].d\vec{l} - \rho g dz$$

$$\implies dp = -\rho\frac{\partial \vec{q}}{\partial t}.d\vec{l} - \frac{\rho}{2}\nabla(\vec{q}.\vec{q}).d\vec{l} + \rho\{\vec{q}\times\vec{\Omega}\}.d\vec{l} - \rho g dz$$

***Note:*** Let us write the simplified version of the second term of above equation;

$$\frac{\rho}{2}\nabla(\vec{q}.\vec{q}).d\vec{l} = \frac{\rho}{2}\left[\hat{i}\frac{\partial \vec{q}^2}{\partial x} + \hat{j}\frac{\partial \vec{q}^2}{\partial y} + \hat{k}\frac{\partial \vec{q}^2}{\partial z}\right].d\vec{l}$$

$$\implies \frac{\rho}{2}\nabla(\vec{q}.\vec{q}).d\vec{l} = \frac{\rho}{2}\left[\hat{i}\frac{\partial \vec{q}^2}{\partial x} + \hat{j}\frac{\partial \vec{q}^2}{\partial y} + \hat{k}\frac{\partial \vec{q}^2}{\partial z}\right].(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\implies \frac{\rho}{2}\nabla(\vec{q}.\vec{q}).d\vec{l} = \frac{\rho}{2}\left[\hat{i}\frac{\partial \vec{q}^2}{\partial x}dx + \hat{j}\frac{\partial \vec{q}^2}{\partial y}dy + \hat{k}\frac{\partial \vec{q}^2}{\partial z}dz\right]$$

$$\implies \frac{\rho}{2}\nabla(\vec{q}.\vec{q}).d\vec{l} = \frac{\rho}{2}d\vec{q}^2.$$

Therefore we have;

$$
\begin{aligned}
dp &= -\rho\frac{\partial\vec{q}}{\partial t}.\vec{dl} - \frac{\rho}{2}d\vec{q}^2 + \rho(\vec{q}\times\vec{\Omega}).\vec{dl} - \rho g dz \\
\implies dp &+ \frac{\rho}{2}d\vec{q}^2 + \rho g dz + \rho\frac{\partial\vec{q}}{\partial t}.\vec{dl} + \rho[\vec{dl}.(\vec{q}\times\vec{\Omega})] = 0
\end{aligned}
$$

> ***Note:***
>
> - ***If velocity of the fluid element is in direction of*** $\vec{dl}$ ***i.e.*** $\vec{dl}$ ***is along the streamlines*** then;
>
> $$[\vec{dl}.(\vec{q}\times\vec{\Omega})] = \begin{vmatrix} dx & dy & dz \\ u & v & z \\ \Omega_x & \Omega_y & \Omega_z \end{vmatrix} = 0 \qquad [\because R_1 \text{ is scalar multiple of } R_2]$$
>
> - ***If the flow is steady*** then $\frac{\partial\vec{q}}{\partial t} = 0$.

Therefore, we have;

$$
\begin{aligned}
dp &+ \frac{\rho}{2}d\vec{q}^2 + \rho g dz + \rho\cancelto{0}{\frac{\partial\vec{q}}{\partial t}}.\vec{dl} + \rho\cancelto{0}{[\vec{dl}.(\vec{q}\times\vec{\Omega})]} = 0 \\
\implies dp &+ \frac{1}{2}\rho d\vec{q}^2 + \rho g dz = 0. \qquad (10)
\end{aligned}
$$

This (above) equation is the required ***Euler's equation along a streamline*** *or* ***Bernoulli's equation of motion***.

***If*** $\rho$ ***is constant:*** then from above equation (10), we can write;

$$
\begin{aligned}
\int_P^Q \frac{dp}{\rho} + \frac{1}{2}\int_P^Q d\vec{q}^2 + \int_P^Q g dz = 0 \\
\frac{1}{\rho}(p_Q - p_P) + \frac{1}{2}(\vec{q}_Q^2 - \vec{q}_P^2) + g(z_Q - z_P) = 0 \qquad (11)
\end{aligned}
$$

> ***Note:*** As a conclusion we can say that *if the flow is irrotational then Bernoulli's equation can be applied between any two points P and Q in the flow field.*

### 3.2.1 Unsteady Version

If $\vec{dl}$ is along the streamlines but the ***flow is unsteady***, then we have;

$$
\begin{aligned}
dp &+ \frac{\rho}{2}d\vec{q}^2 + \rho g dz + \rho\frac{\partial\vec{q}}{\partial t}.\vec{dl} + \rho\cancelto{0}{[\vec{dl}.(\vec{q}\times\vec{\Omega})]} = 0 \\
\implies dp &+ \frac{\rho}{2}d\vec{q}^2 + \rho g dz + \rho\frac{\partial\vec{q}}{\partial t}.\vec{dl} = 0 \\
\implies dp &+ \frac{\rho}{2}d\vec{q}^2 + \rho g dz = -\rho\frac{\partial q}{\partial t}ds
\end{aligned}
$$

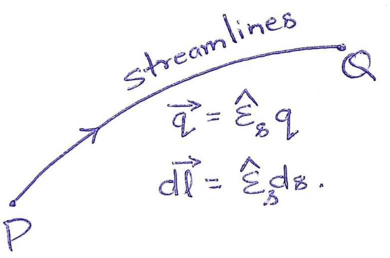


Figure 5: depicting the considered streamline for Euler's equation.

If $\rho$ is constant, then we have;

$$\frac{1}{\rho}(p_Q - p_P) + \frac{1}{2}(\vec{q}_Q^2 - \vec{q}_P^2) + g(z_Q - z_P) = -\int_P^Q \frac{\partial q}{\partial t} ds$$
$$\implies \frac{p_Q}{\rho} + \frac{\vec{q}_Q^2}{2} + gz_Q = \frac{p_P}{\rho} + \frac{\vec{q}_P^2}{2} + gz_P - \int_P^Q \frac{\partial q}{\partial t} ds$$

***Note:*** ***If flow is irrotational***, then $\implies$ $\vec{q} \times \vec{\Omega} = 0$ and this further $\implies$ $\exists$ a scalar function $\phi$ such that $\vec{q} = \nabla.\phi$ and we get the ***Bernoulli's equation*** as;

$$\frac{dp}{\rho} + \frac{1}{2} d(\nabla\phi^2) + gdz = -d\left(\frac{\partial\phi}{\partial t}\right). \tag{12}$$

This (above) equation is the ***Euler's equation in terms of velocity potential***.

✠✠✠✠✠