Lecture notes on mathematics

Dr. Sunil Kumar
Associate professor

Department of Mathematics,
National Institute of Technology, Jamshedpur.

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Outlines

Consider the differential equation with variable coefficients

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{1}$$

where $P_0(x)$, $P_1(x)$ and $P_2(x)$ are polynomial in x.

(a). Ordinary point-: A point $x = a$ is called ordinary point of the above differential equation (DE) if $P_0(x) \neq 0$ at $x = a$.

(b). Singular point-: A point $x = a$ is called Singular point of the above differential equation (DE) if $P_0(x) = 0$ at $x = a$.

(i). Regular Singular point-: If $\frac{P_1(x)}{P_0(x)}$ and $\frac{P_2(x)}{P_0(x)}$ are differentiable in the neighborhood of $x = a$.

(ii). Irregular Singular point-: If not regular then irregular singular point.

Working Rule:-

I. In the given DE (1), now $P_0(x) \neq 0$ at $x = 0$ is the ordinary point.

II. Consider the series solution of DE (1)

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \tag{2}$$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

III. Put y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the DE (1) and simplifying it. Now, equating the coefficients of x^0, x^1, x^2, ..., x^n to zero. It gives the values of a_2,a_3,a_4,...,a_n in terms of a_0 and a_1.

IV. Put the values of a_2, a_3, a_4,...,a_n into (2) we get the solution.

When $x = 0$ is the regular singular point of the DE (Frobenius method).
Working Rule:-
I. Given DE

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{3}$$

$x = 0$ is the regular singular point.
II. Let $y = x^m(a_0 + a_1x + a_2x + ... + a_nx^n + ...)$ be the series solution of (3). Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
II. Put y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into (3) and simplify it. Equating the coefficient of lowest degree of x to zero, we obtain a quadratic equation in m. This equation is called indical equation. Solve indical equation we get two roots m_1 and m_2 (say). Moreover, the complete solution depends on the nature of roots.
Case-I. If $m_1 \neq m_2$ and $m_1 - m_2 \neq$ an integer then the complete solution is

$$y = C_1(y)_{m=m_1} + C_2(y)_{m=m_2}$$

Case-II. If $m_1 = m_2$ then the complete solution is

$$y = C_1(y)_{m_1} + C_2(\frac{\partial y}{\partial m})_{m_1}.$$

Case-III. If $m_1 \neq m_2$ and $m_1 - m_2 =$ an integer, then the complete solution is

$$y = C_1(y)_{m=m_1} + C_2(\frac{\partial y}{\partial m})_{m_2}, \quad (m_1 > m_2).$$

Assignment

Q.1 Solve $2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x+1)y = 0.$

Q.2 Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$

Legendre differential equation

Legendre differential equation is defined as

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha+1)y = 0 \tag{4}$$

where n is positive integer.

The Legendre equation can be put in the form

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha+1)}{1-x^2}, \quad \text{if } x^2 \neq 1.$$

Since $\frac{1}{(1-x^2)} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$, both $p(x)$ and $q(x)$ have power series expansions in the open interval $(-1, 1)$.

Thus, seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-1, 1).$$

Differentiating term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Thus,

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = \sum_{n=0}^{\infty} 2n a_n x^n,$$

and

$$\begin{aligned}
(1-x^2)y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n(n-1) a_n] x^n.
\end{aligned}$$

On substituting it into the equation (4) we obtain

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n = 0, \quad n \geq 0,$$

which leads to a recurrence relation

$$a_{n+2} = -\frac{(\alpha - n)(\alpha + n + 1)}{(n+1)(n+2)} a_n.$$

Thus, we obtain

$$\begin{aligned} a_2 &= -\frac{\alpha(\alpha+1)}{1 \cdot 2} a_0, \\ a_4 &= -\frac{(\alpha-2)(\alpha+3)}{3 \cdot 4} a_2 = (-1)^2 \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!} a_0, \\ &\vdots \\ a_{2n} &= (-1)^n \frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!} a_0. \end{aligned}$$

Similarly, we can compute $a_3, a_5, a_7, \ldots,$ in terms of a_1 and obtain

$$\begin{aligned} a_3 &= -\frac{(\alpha-1)(\alpha+2)}{2\cdot 3}a_1 \\ a_5 &= -\frac{(\alpha-3)(\alpha+4)}{4\cdot 5}a_3 = (-1)^2\frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}a_1 \\ &\vdots \\ a_{2n+1} &= (-1)^n\frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!}a_1 \end{aligned}$$

Therefore, the series for $y(x)$ can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x), \quad \text{where}$$

$y_1(x) = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!}x^{2n}$, and
$y_2(x) = x + \sum_{n=1}^{\infty}(-1)^n\frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)\cdot(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!}x^{2n+1}$.

Bessel's differential equation is defined as

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0. \tag{5}$$

Or,

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+(1-\frac{n^2}{x^2})y=0. \tag{6}$$

Since, $x=0$ is regular singular point. Let the series solution is $y=\sum_{r=0}^{\infty}a_r x^{k+r}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and put it into equation (5)

- **Recurrence Relations of Bessel's Function**

1. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Proof

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)}$$

$$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{x^{2n+2r}}{2^{n+2r}}\frac{1}{r!\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r\frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}}\frac{1}{r!(n+r)\Gamma(n+r)}$$

$$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = x^n\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{(n-1)+2r}\frac{1}{r!\Gamma((n-1)+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = x^n J_n(x). \qquad \square$$

2. $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

Proof

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)}$$

$$\Rightarrow x^{-n}J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{n+2r}}\frac{1}{r!\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r\frac{2rx^{2r-1}}{2^{n+2r}}\frac{1}{(r-1)!r\Gamma(n+r)}$$

$$= x^{-n}\sum_{r=1}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{(r-1)!\Gamma(n+r+1)}$$

$$= x^{-n}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{(n+1)+2k}\frac{1}{k!\Gamma((n+1)+k+1)}$$

Putting $r = k+1$

$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = x^{-n}J_{n+1}(x). \qquad \square$$

3. $J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

Proof

From recurrence relation (1)

$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$\Rightarrow x^n J_n'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)$

Dividing by x^n, we get

$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$

$\Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x).$

4. $J_n'(x) = -J_{n-1}(x) + \frac{n}{x}J_n(x)$

Proof

From recurrence relation (2)

$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

$\Rightarrow x^{-n} J_n'(x) - nx^{-n-1}J_n(x) = -x^n J_{n+1}(x)$

Dividing by x^{-n}, we get

$\Rightarrow J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x).$

5. $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Proof

Adding recurrence relation (3) and (4), we get
$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

6. $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Proof

Subtracting recurrence relation (3) and (4), we get
$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}$
$\Rightarrow 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$.

The function $(1-2xz+z^2)^{-\frac{1}{2}}$ is called the generating function of Legendre's ploynomials as $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n$

Proof: $(1-2xz+z^2)^{-\frac{1}{2}} = [1-(2xz-z^2)]^{-\frac{1}{2}}$

$= 1 + \frac{1}{2}(2xz-z^2) + \frac{1}{2}\frac{3}{4}(2xz-z^2)^2 + \ldots + \frac{1}{2}\frac{3}{4}\ldots\frac{2k-1}{2}(2xz-z^2)^k + \ldots$

$\because (1-t)^{-\frac{1}{2}} = 1 + \frac{1}{2}t + \frac{1}{2}\frac{3}{2}\frac{t^2}{2!} + \ldots$

$= 1 + \sum_{k=0}^{\infty} \frac{1}{2}\frac{3}{4}\ldots\frac{2k-1}{2k}(2xz-z^2)^k, \qquad \ldots.(1)$

Again $(2xz-z^2)^k = z^k[2x-z]^k$

$= z^k[(2x)^k - k(2x)^{k-1}z + \frac{k(k-1)}{2!}(2x)^{k-2}z^2 - \ldots + (-1)^k z^k], \qquad \ldots.(2)$

Using (2) in (1) we get,

$(1-2xz+z^2)^{-\frac{1}{2}} =$

$1+\sum_{k=0}^{\infty} \frac{1}{2}\frac{3}{4}\ldots\frac{2k-1}{2k}[(2x)^k z^k - k(2x)^{k-1}z^{k+1} + \frac{k(k-1)}{2!}(2x)^{k-2}z^{k+2} - \ldots + (-1)^k z^{2k}], \qquad \ldots.(3)$

Coefficient of z^n in expression (3) is given by

$\frac{1}{2}\frac{3}{4}\ldots\frac{2n-1}{2n}(2x)^n - \frac{1}{2}\frac{3}{4}\ldots\frac{(2n-3)}{(2n-2)}(n-1)(2x)^{n-2} + \frac{1}{2}\frac{3}{4}\ldots\frac{(2n-5)}{(2n-4)}\frac{(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots$

$= \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\right]$

$= P_n(x)$

$\therefore (1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n. \qquad \square$

- **Recurrence Relations of Legendre's Function**

1. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Proof: From generating function

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x) \tag{7}$$

Differentiating both side of (7) partially with respect to z, we get

$-\frac{1}{2}(1 - 2xz + z^2)^{-\frac{3}{2}}(-2x + 2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x - z)(1 - 2xz + z^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x - z)(1 - 2xz + z^2)^{-\frac{1}{2}} = (1 - 2xz + z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x - z)\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$, by using (7)

Equating coefficient of z^n on both side

$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

2. $P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$

Proof: Differentiating recurrence relation (7) partially with respect to x, we get

$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2z) = \sum_{n=0}^{\infty} z^n P'_n(x)$

$\Rightarrow z(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n P'_n(x)$

$\Rightarrow z\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n P'_n(x)$, by using (7)

Equating coefficient of z^{n+1} on both side

$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x).$

3. $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

Proof: Differentiating recurrence relation (7) partially with respect to x, we get

$(n+1)P'_{n+1}(x) = (2n+1)xP'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x), \qquad \ldots(a)$

Also from recurrence relation (2)

$P'_{n+1}(x) = P_n(x) + 2xP'_n(x) - P'_{n-1}(x), \qquad \ldots(b)$

Using (a) and (b) , we get

$(n+1)[P_n(x) + 2xP'n(x) - P'_{n-1}(x)] = (2n+1)xP'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x)$

$\Rightarrow nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

4. $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$

Proof: Adding recurrence relation (2) and (3), we get
$(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$.

5. $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Proof: Adding recurrence relation (3) and (4), we get
$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

6. $(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

Proof: Replacing n by (n-1) in recurrence relation (4)
$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x) \qquad \ldots(c)$
Also multiplying recurrence relation (3) by x
$nxP_n(x) = x^2P'_n(x) - xP'_{n-1}(x) \qquad \ldots(d)$
Subtracting (d) from (c)
$(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

Assignment

- Prove following

(1) $P_n(1) = 1.$ (2) $P_n(-1) = (-1)^n.$

■**Orthogonality of Legendre's ploynomial**

Orthogonality property of Legendre's polynomials is given by the relations

$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$, when $m \neq n$

and $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$, when $m = n$.

where m and n are positive integers.

■**Orthogonality of Bessel's function**

If α and β be the roots of $J_n(x) = 0$, then

$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0$, if $\alpha \neq \beta$

and

$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{1}{2}J_{n+1}^2(\alpha)$, if $\alpha = \beta$.

where m and n are positive integers.