Numerical Methods and Application

Code: MA 1404

Ramayan Singh

Department of Mathematics
NIT Jamshedpur

Contents

Topic No. 1

Introduction to Operator

Learning Objectives At the end of this session students should be able to

1. recall forward and bacward difference operator.
2. know what is central diffetence and average operator.
3. know what is shift operator.
4. establish the relationship between above five operator.

THEORITICAL PART

1.1 Forward difference operator

Forward difference operator , denoted by Δ, and defined by

$$\Delta f(x) = f(x+h) - f(x).$$

The expression $f(x+h) - f(x)$ gives the **First Forward difference** of $f(x)$ and the operator Δ is called the **First Forward difference operator**. Given the step size h, this formula uses the values at x and $x+h$, the point at the next step. As it is moving in the forward direction, it is called the forward difference operator.

Similarly, the second forward difference operator, Δ^2, is defined as

$$\Delta^2 f(x) = \Delta\big(\Delta f(x)\big) = \Delta f(x+h) - \Delta f(x).$$

We note that

$$\begin{aligned} \Delta^2 f(x) &= \Delta f(x+h) - \Delta f(x) \\ &= \big(f(x+2h) - f(x+h)\big) - \big(f(x+h) - f(x)\big) \\ &= f(x+2h) - 2f(x+h) + f(x) \end{aligned}$$

In particular, for $x = x_k$, we get,

$$\Delta y_k = y_{k+1} - y_k$$

and

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k = y_{k+2} - 2y_{k+1} + y_k.$$

Now the r^{th} forward difference operator, Δ^r, is defined as

$$\Delta^r f(x) = \Delta^{r-1} f(x+h) - \Delta^{r-1} f(x), \qquad r = 1, 2, \ldots,$$

$$\text{with } \Delta^0 f(x) = f(x).$$

Remark:

- For a set of tabular values, the horizontal forward difference table is written as:

x_0	y_0		
		$\Delta y_0 = y_1 - y_0$	
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$
		$\Delta y_1 = y_2 - y_1$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$
$\ldots$	$\ldots$	$\ldots$	
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$
		$\Delta y_{n-1} = y_n - y_{n-1}$	
x_n	y_n		

- In general, if $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n$ is a polynomial of degree n, then it can be shown that $\Delta^n f(x) = n!\, h^n$ and $\Delta^{n+r} f(x) = 0$ for $r = 1, 2, \ldots$.

1.2 Backward difference operator

The Backward difference operator, denoted by ∇, is defined as

$$\nabla f(x) = f(x) - f(x-h).$$

Given the step size h, note that this formula uses the values at x and $x-h$, the point at the previous step. As it moves in the backward direction, it is called the backward difference operator.

The r^{th} **backward difference operator**, ∇^r, is defined as

$$\nabla^r f(x) = \nabla^{r-1} f(x) - \nabla^{r-1} f(x-h), \qquad r = 1, 2, \ldots,$$

$$\text{with } \nabla^0 f(x) = f(x).$$

In particular, for $x = x_k$, we get $\nabla y_k = y_k - y_{k-1}$ and $\nabla^2 y_k = y_k - 2y_{k-1} + y_{k-2}$.

Note: $\nabla^2 y_k = \Delta^2 y_{k-2}$.

1.3 Central difference operator

The The First **Central difference operator**, denoted by δ, is defined by

$$\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$$

and the r^{th} Central difference operator is defined as

$$\delta^r f(x) = \delta^{r-1} f(x + \frac{h}{2}) - \delta^{r-1} f(x - \frac{h}{2})$$

$$\text{with } \delta^0 f(x) = f(x).$$

Thus, $\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h)$.

In particular, for $x = x_k$, define $y_{k+\frac{1}{2}} = f(x_k + \frac{h}{2})$, and $y_{k-\frac{1}{2}} = f(x_k - \frac{h}{2})$, then

$\delta y_k = y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}$ and $\delta^2 y_k = y_{k+1} - 2y_k + y_{k-1}$.

Example 1.3.1. the following set of tabular values (x_i, y_i), write the forward and backward difference tables.

x_i	9	10	11	12	13	14
y_i	5.0	5.4	6.0	6.8	7.5	8.1

Soln.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
9	5.0	0.4	0.2	0.0	-0.3	0.6
10	5.4	0.6	0.2	-0.3	0.3	
11	6.0	0.8	-0.1	0.0		
12	6.8	0.7	-0.1			
13	7.5	0.6				
14	8.1					

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
9	5					
10	5.4	0.4				
11	6	0.6	0.2			
12	6.8	0.8	0.2	0.0		
13	7.5	0.7	-0.1	- 0.3	-0.3	
14	8.1	0.6	-0.1	0.0	0.3	0.6

1.4 Shift Operator

A Shift operator, denoted by E, is the operator which shifts the value at the next point with step h, i.e.,

$$Ef(x) = f(x+h).$$

Thus,

$$Ey_i = y_{i+1}, \quad E^2 y_i = y_{i+2}, \quad \text{and} \quad E^k y_i = y_{i+k}.$$

Example 1.4.1. Show that $E \equiv 1 + \Delta$

Soln.

$$\begin{aligned} Ef(x) &= f(x+h) \\ &= [f(x+h) - f(x)] + f(x) \\ &= \Delta f(x) + f(x) \\ &= (\Delta + 1) f(x) \end{aligned}$$

$$\therefore E \equiv 1 + \Delta$$

1.5 Averaging Operator

The **Averaging Operator**, denoted by μ, gives the average value between two central points, i.e., $\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})]$. Thus $\mu y_i = \frac{1}{2}(y_{i+\frac{1}{2}} + y_{i-\frac{1}{2}})$ and $\mu^2 y_i = \frac{1}{2}\left[\mu y_{i+\frac{1}{2}} + \mu y_{i-\frac{1}{2}}\right] = \frac{1}{4}\left[y_{i+1} + 2y_i + y_{i-1}\right]$.

Example 1.5.1. Show that $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$.

Soln.

Let us denote by $E^{\frac{1}{2}} f(x) = f(x+\frac{h}{2})$

Then, we see that

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$$
$$= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$
$$\therefore \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Example 1.5.2.

Estimate the missing term from the following table:

x_i	0	1	2	3	4	5
y_i	0	–	8	15	–	35

Soln.

Since we are given four values, therefore we take y to be a polynomial of degree 3 in x so that

$$\Delta^4 y = 0 \quad \text{for all values of } x$$
$$\therefore \quad (E-1)^4 y = 0$$
$$\text{i.e.,} \quad E^4 y - 4E^3 y + 6E^2 y - 4Ey + y = 0$$

Putting $x = 0$, we get

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$
$$\text{i.e., } y_4 - 4y_1 = 12 \tag{1}$$

Again putting $x = 1$, we get

$$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0$$
$$\text{i.e., } 4y_4 - y_1 = 93 \tag{2}$$

solving (1) and (2) we get $y_1 = 3, y_4 = 24$

Practice Problems

Problem 1.6. 1. Prove that $E \equiv e^{hD}$.

2. Prove that $\mu^2 \equiv \frac{1}{4}\left(\delta^2 + 4\right)$

3. Prove that $hD \equiv \sinh^{-1}(\mu\delta)$.

4. Prove that $\Delta - \nabla \equiv \delta^2$

5. Show that $D \equiv \frac{1}{h} \ln\left(\frac{1}{1-\nabla}\right)$.

6. Find $\Delta^2(ax^2 + bx + c)$ *Ans:* $2ah^2$

7. Find $\Delta^2 \cos 2x$ *Ans:* $-4 \sin h \cos 2(x + h)$

8. Estimate the missing term from the following table:

x_i	0	1	2	3	4
y_i	1	3	9	–	81

Ans: 31

9. Estimate the missing term from the following table:

x_i	1	2	3	4	5	6	7
y_i	2	4	8	–	32	64	128

Ans: 16.1

Topic No. 2

Interpolation

Learning Objectives At the end of this session students should be able to

1. know what is interpolation.
2. apply Lagrange method and divide difference method to find the unknown value from the given data set.
3. apply Newton forward and backward interpolation method.

THEORITICAL PART

Interpolation is a method of constructing new data points within the range of a discrete set of known data points.

In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate, i.e., estimate the value of that function for an intermediate value of the independent variable.

2.1 Lagrange Interpolation

Let us suppose that the given data points (x_i, y_i), $i = 0, 1, 2...n$ is coming from a function $f(x)$. Let us assume that this function $y = f(x)$ takes the values $y_0, y_1.......y_n$ at $x_0, x_1, ...x_n$. Since there are $(n+1)$ data points (x_i, y_i), we can represent the function $f(x)$ by a polynomial of degree n

$$\therefore f(x) = C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0 \tag{3}$$

As we have assumed that $f(x_i) = y_i, i = 0, 1, 2.......n$ i.e. the function $f(x)$ passes through (x_i, y_i) can be rewritten as:

$$\begin{aligned} y = f(x) = a_0(x - x_1)(x - x_2)...(x - x_n) + a_1(x - x_0)(x - x_2)...(x - x_n) + \\ a_2(x - x_0)(x - x_1)(x - x_3)...(x - x_n) + ... + a_n(x - x_0)...(x - x_{n-1}) \end{aligned} \tag{4}$$

But

$$y_i = f(x_i) \quad i = 0, 1, ...n \tag{5}$$

Using (3) for i=0, in (2) we get

$$y_0 = f(x_0) = a_0(x_0 - x_1)...(x_0 - x_n) \tag{6}$$

$$\therefore a_0 = \frac{y_0}{(x_0 - x_1)...(x_0 - x_n)} \tag{7}$$

For $i = 1$, we get

$$y_1 = f(x_1) = a_1(x_1 - x_o)(x_1 - x_2)...(x_1 - x_n) \tag{8}$$

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)} \tag{9}$$

Similarly for $i = 2.......n - 1$, we get

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)} \tag{10}$$

and for $i = n$, we get

$$a_n = \frac{y_n}{(x_n - x_0)...(x_n - x_{n-1})} \tag{11}$$

Using (5), (7), (8), (9) in (2) we get can be rewritten in a compact form as:

$$\begin{aligned} y = f(x) &= L_0(x)y_0 + L_1(x)y_1 + ... + L_n(x)y_n \\ &= \sum_{i=0}^{n} L_i(x)y_i \\ &= \sum_{i=0}^{n} L_i(x)f(x_i) \end{aligned} \tag{12}$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x......)}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)} \tag{13}$$

It can be easily noted that Let us introduce the product notation as :

$$\prod(x) = \prod_{i=0}^{n}(x - x_i) = (x - x_0)(x - x_1)....(x - x_n)$$

Therefore, Lagrange interpolation polynomial of degree n can be written as

$$y = f(x) = \sum_{k=0}^{n} L_k(x)y_k \tag{14}$$

Note: Given a set of data points(x_i, y_i) $i = 1, 2, \cdots, n$. Suppose we are interested in evaluating $f(x)$ at some intermediate point x to a desired level of accuracy. Directly using the entire data set of size n may not only be computationally economical but may also turn out to be redundant. Naturally one would like to use an interpolating polynomial of optimal degree. Since this is not known apriori, one may start with $P_0(x)$ and if it was enough then move onto $P_1(x)$ and so on i.e. slowly increase the no. of the interpolating points (or) data points $x_0, x_1..x_k$ so that $P_{k-1}(x)$ will be close to $f(x)$. In this context the biggest disadvantage with Lagrange Interpolation is that we cannot use the work that has already been done i.e. we cannot make use of $P_{k-1}(x)$ while evaluating $P_k(x)$. With the addition of each new data point, calculations have to be repeated. Newton Interpolation polynomial overcomes this drawback.

Example 2.1.1. Given the following data table, construct the Lagrange interpolation polynomial $f(x)$, to fit the data and find $f(1.25)$

x_i	0	1	2	3
y_i	1	2.25	3.75	4.25

Soln. Lagrange interpolation polynomial is given by y=$f(x)$ =

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{x^3-6x^2+11x-6}{-6}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{x^3-5x^2+6x}{2}$$

$$L_2(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{x^3-4x^2+3x}{-2}$$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{x^3-3x^2+2x}{6}$$

$$\begin{aligned} f(1.25) &= L_0(1.25)y_0 + L_1(1.25)y_1 + L_2(1.25)y_2 + L_3(1.25)y_3 \\ &= (-0.546875).1 + (0.8203125)2.25 + (0.2734375)3.75 + (-0.0390625)4.25 \\ &= 2.650390625 \end{aligned}$$

Example 2.1.2. Given the following data table, construct the Lagrange interpolation polynomial $f(x)$, to fit the data and find

x_i	1980	1985	1990	1995	2000	2005
y_i	440	510	525	571	500	600

Soln. Here $n = 6$, $x_k = 1998$

Lagrange interpolation polynomial is given by

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} \\ &= \frac{(x-1985)(x-1990)(x-1995)(x-2000)(x-2005)}{(1980-1985)(1980-1990)(1980-1995)(1980-2000)(1980-2005)} \\ L_0(x_k) = L_0(1998) &= \frac{(1998-1985)(1998-1990)(1998-1995)(1998-2000)(1998-2005)}{(-5)(-10)(-15)(-20)(-25)} \\ &= \frac{13.8.3.(-2).(-7)}{-(375000)} = -\frac{4368}{375000} = -0.011648 \\ L_1(x_k) &= \frac{(x_k-x_0)(x_k-x_2)(x_k-x_3)(x_k-x_4)(x_k-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} \\ &= \frac{(1998-1980)(1998-1990)(1998-1995)(1998-2000)(1998-2005)}{(1985-1980)(1985-1990)(1985-1995)(1985-2000)(1985-2005)} \\ &= \frac{18.8.3.(-2).(-7)}{5(-5)(-10)(-15)(-20)} = 0.08064 \end{aligned}$$

$$L_2(x_k) = \frac{(1998-1980)(1998-1985)(1998-1995)(1998-2000)(1998-2005)}{(1990-1980)(1990-1985)(1990-1995)(1990-2000)(1990-2005)}$$
$$= \frac{18.13.3.(-2)(-7)}{10.5.(-5)(-10).(-15)} = -0.26208$$

$$L_3(x_k) = \frac{(1998-1980)(1998-1985)(1998-1990)(1998-2000)(1998-2005)}{(1995-1980)(1995-1985)(1995-1990)(1995-2000)(1995-2005)}$$
$$= \frac{18.13.8.(-2)(-7)}{15.10.5(-5)(-10)} = 0.69888$$
$$L_4(x_k) = \frac{(1998-1980)(1998-1985)(1998-1990)(1998-1995)(1998-2005)}{(2000-1980)(2000-1985)(2000-1990)(2000-1995)(2000-2005)}$$
$$= \frac{18.13.8.3.(-7)}{20.15.10.5(-5)} = 0.52416$$
$$L_4(x_k) = \frac{(1998-1980(1998-1985)(1998-1990)(1998-1995)(1998-2000)}{(2005-1980)(2005-1985)(2005-1990)(2005-1995)(2005-2000)}$$
$$= \frac{18.13.8.3.(-2)}{25.20.15.10.5} = -0.029952$$

$$\therefore \qquad f(1998) = \sum_{i=0}^{5} L_i(1998)y_i = 541.578560$$

Practice Problems

Problem 2.2. 1. Use Lagrange's interpolation formula to find $f(x)$, if $f(1) = 2,\ f(2) = 4, f(3) = 8, f(4) = 16$ and$f(7) = 128$.

2. Use Lagrange interpolation formula to fit a polynomial to the following data. Hence find $f(4)$.

x_i	1.140	1.145	1.150	1.155	1.160	1.165
$f(x)$	0.13103	0.135410	0.13976	0.14410	1.14842	0.15272

Ans: $1/6(7x^3 - 31x^2 + 28x + 18)$,13.66

2.3 Newton Interpolation polynomial

Suppose that we are given a data set $(x_i, f_i), i = 0, 1...n-1$. Let us assume that these are interpolating points of Newton form of interpolating polynomial $P_n(x)$ of degree n i.e

$$P_n(x_i) = f_i, \quad i = 1, 2, \cdots, n \tag{15}$$

The Newton form of the interpolating polynomial $P_n(x)$ is given by

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_n) \tag{16}$$

For $i = 0$, from (13) & (14) we get

$$f_0 = P_n(x_0) = a_0$$

For $i = 1$, from (1) & (2) we get

$$f_1 = p_n(x_1) = a_0 + a_1(x_1 - x_0)$$
$$a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

For $i = 2$, from(13) & (14) we get

$$f_2 = p_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

Using a_0 & a_1 we get

$$a_2 = \frac{[(f_2 - f_1)/(x_2 - x_1)] - [(f_1 - f_0)/(x_1 - x_0)]}{(x_2 - x_0)}$$

Similarly we can find $a_3.....a_{n-1}$. To express $a_i, i = 0......n-1$ in a compact manner let us first define the following notation called divided differences:

$$f[x_k] = f_k$$
$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}$$
$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

Now the co-efficients can be expressed in terms of divided differences as follows:

$$a_0 = f_0 = f[x_0]$$
$$a_2 = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$
$$= f[x_0, x_1, x_2]$$

$$a_n = f[x_0, x_1 ... x_n]$$

Note that a_1 is called as the first divided difference, a_2 as the second divided difference and so on. Now the polynomial (13) can be rewritten as:

This is called as Newton's Divided Difference interpolation polynomial.

Example 2.3.1. Given the following data table, evaluate $f(2.4)$ using 3^{rd} order Newton's Divided Difference interpolation polynomial.

x_i	0	1	2	3	4
$y_i = f(x_i)$	1	2.25	3.75	4.25	5.81

Soln. Here . For constructing 3^{rd} order Newton Divided Difference polynomial we need only four points. Let us use the first four points. The 3^{rd} Newton Divided Difference polynomial is given by:

$$
\begin{aligned}
a_0 &= f[x_0] = 1\\
\therefore a_1 = f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{2.25 - 1}{1 - 0} = 1.25\\
\therefore a_2 = f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.5 - 1.25}{2 - 0} = 0.125\\
f[x_2, x_3] &= \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{4.25 - 3.75}{3 - 2} = \frac{0.5}{1} = 0.5\\
f[x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{0.5 - 1.5}{3 - 1} = -0.5\\
\therefore a_3 = f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.5 - 0.125}{3 - 0} = \frac{-0.625}{3} = -0.20833
\end{aligned}
$$

$$
\begin{aligned}
\therefore p_3(x) &= 1 + 1.25(x - 0) + 0.125(x - 0)(x - 1) + (-0.20833)(x - 0)(x - 1)(x - 2)\\
\therefore f(2.4) = p_3(2.4) &= 1 + 1.25(2.4 - 0) + 0.125(2.4 - 0)(2.4 - 1)\\
&+ (-0.20833)(2.4 - 0)(2.4 - 1)(2.4 - 2)\\
&= 1 + (1.25)(2.4) + 0.125(2.4)(1.4) - 0.20833(2.4)(1.4)(0.4)
\end{aligned}
$$

In this example it may be noted that for calculating the order polynomial, we first start with $P_0 = f[x_0] = 1$. To it we add $a_1(x - x_0)$ to get P_1 and to P_1 we add $a_2(x - x_0)(x - x_1)$ to get P_2. Finally on adding $a_3(x - x_0)(x - x_1)(x - x_2)$ to P_2 we get P_3.

Newton Divided Difference Table

It may also be noted for calculating the higher order divided differences we have used lower order divided differences. In fact starting from the given zeroth order differences ; one can systematically arrive at any of higher order divided differences. For clarity the entire calculation may be depicted in the form of a table called

i	x_i	$f[x_i]$	First order differences	Second order differences	Third order differences	Fourth order differences
0	x_0	$f[x_0]$				
			$f[x_0, x_1]$			
1	x_1	$f[x_0]$		$f[x_0, x_1, x_2]$		
			$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
2	x_2	$f[x_2]$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
			$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
3	x_3	$f[x_3]$		$f[x_2, x_3, x_4]$		
			$f[x_3, x_4]$			
4	x_4	$f[x_4]$				

Again suppose that we are given the data set (x_i, f_i), $i = 0.......5$ and that we are interested in finding the 5^{th} order Newton Divided Difference interpolation polynomial. Let us first construct the Newton Divided Difference Table. Wherein one can clearly see how the lower order differences are used in calculating the higher order Divided Differences

Note: One may note that the given data corresponds to the cubic polynomial. To fit such a data 3^{rd} order polynomial is adequate. From the Newton Divided Difference table we notice that the fourth order difference is zero. Further the divided differences in the table can be directly used for constructing the Newton Divided Difference interpolation polynomial that would fit the data.

Example 2.3.2. Construct the Newton Divided Difference Table for generating Newton interpolation polynomial with the following data set:

x_i	0	1	2	3	4
$y_i = f(x_i)$	0	1	8	27	64

Soln. Here . One can fit a fourth order Newton Divided Difference interpolation polynomial to the given data. Let us generate Newton Divided Difference Table; as requested.

i	x_i	$f[x_i]$	1^{st} order differences	2^{nd} order differences	3^{rd} order differences	4^{th} order differences
0	0	0				
			$\frac{1-0}{1-0} = 1$			
1	1	1		$\frac{7-1}{2-0} = 3$		
			$\frac{8-1}{2-1} = 7$		$\frac{6-3}{3-0} = 1$	
2	2	8		$\frac{19-7}{3-1} = 6$		$\frac{1-1}{4-0} = 0$

			$\frac{27-8}{3-2}=19$		$\frac{9-6}{4-1}=1$
3	3	27		$\frac{37-19}{4-2}=9$	
			$\frac{64-27}{4-3}=37$		
4	4	64			

Practice Problems

Problem 2.4. 1. Find $f(8)$ using Newton's divided difference formula given that.

x_i	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Ans: 448

2. Find $f(27)$ using Newton's divided difference formula given that.

x_i	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

Ans: 49.3

2.5 Gregory-Newton Forward Difference Approach:

Very often it so happens in practice that the given data set correspond to a sequence $\{x_i\}$ of equally spaced points. Here we can assume that

$$x_i = x_0 + ih \quad i = 0, 1, 2, \cdots, n$$

where x_0 is the starting point (sometimes, for convenience, the middle data point is taken as x_0 and in such a case the integer i is allowed to take both negative and positive values.) and h is the step size. Further it is enough to calculate simple differences rather than the divided differences as in the non-uniformly placed data set case. These simple differences can be forward differences (Δf_i) or backward differences (∇f_i). We will first look at forward differences and the interpolation polynomial based on forward differences.

The first order forward difference Δf_i is defined as

$$\Delta f_i = f_{i+1} - f_i$$

The second order forward difference $\Delta^2 f_i$ is defined as

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

The k^{th} order forward difference $\Delta^k f_i$ is defined as

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$

Since we already know Newton interpolation polynomial in terms of divided differences, to derive or generate Newton interpolation polynomial in terms of forward differences it is enough to express forward differences in terms of divided differences.

Recall the definition of first divided difference $f[x_0, x_1]$,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}$$

$$\therefore \Delta f_0 = hf[x_0, x_1]$$

Similarly we can get

$$\Delta f_1 = hf[x_1, x_2]$$

By the definition of second order forward difference $\Delta^2 f_0$, we get

$$\begin{aligned}\Delta^2 f_0 &= \Delta f_1 - \Delta f_0 \\ &= hf[x_1, x_2] - hf[x_0, x_1] \\ &= h.2h\,(f[x_1, x_2] - f[x_0, x_1]/2h) \\ &= 2h^2 f[x_0, x_1, x_2]\end{aligned}$$

In a similar way, in general, we can show that

$$\Delta^k f_i = k!h^k f[x_i, x_{i+1}, x_{i+2}...x_{i+k}]$$

$$\therefore f[x_i, x_{i+1}, ...x_{i+k}] = \frac{\Delta^k f_i}{k!h^k}$$

For $i = 0$,

$$f[x_0, x_1...x_k] = \frac{\Delta^k f_0}{k!h^k}$$

Now using Newton divide difference formula and above relation the Newton forward difference interpolation polynomial may be written as follows:

$$P_n(x) = \sum_{k=0}^{n} \frac{\Delta^k f_0}{k!h^k} \prod_{i=0}^{k-1}(x - x_i) \tag{17}$$

To rewrite (16) in a simpler way let us set

$$\because x_k = x_0 + kh$$

$$x - x_k = (s - k)h$$

$$
\begin{aligned}
P_n(x) = P_n(s) &= \sum_{k=0}^{n} \frac{\Delta^k f_0}{k!h^k} \prod_{i=0}^{k-1}(s-i)h \\
&= \sum_{k=0}^{n} \frac{\Delta^k f_0}{k!h^k} \quad [s(s-1).......(s-k+1)]h^k \\
&= \sum_{k=0}^{n} \begin{pmatrix} s \\ k \end{pmatrix} \Delta^k f_0 \qquad (18)
\end{aligned}
$$

This is known as Newton-Gregory forward difference interpolation polynomial. For convenience while constructing (17) one can first generate a forward difference table and use the $\Delta^k f_i$ from the table. Suppose we have data set $(x_i f_i)$, $i = 0, 1, 2, 3, 4$ then forward difference table looks as follows:

i	x_i	$f[x_i]$	$\Delta f[x_i]$	$\Delta^2 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^3 f[x_i]$
0	x_0	$f[x_0]$				
			$\Delta f[x_0]$			
1	x_1	$f[x_1]$		$\Delta^2 f[x_0]$		
			$\Delta f[x_1]$		$\Delta^3 f[x_0]$	
2	x_2	$f[x_2]$		$\Delta^2 f[x_1]$		$\Delta^4 f[x_0]$
			$\Delta f[x_2]$		$\Delta^3 f[x_1]$	
3	x_3	$f[x_3]$		$\Delta^2 f[x_2]$		
			$\Delta f[x_3]$			
4	x_4	$f[x_4]$				

2.6 Newton-Gregory Backward Difference Interpolation polynomiale

If the data size is big then the divided difference table will be too long. Suppose the desired intermediate value $(\widetilde{x})$ at which one needs to estimate the function $(i.e. f(\widetilde{x}))$ falls towards the end or say in the second half of the data set then it may be better to start the estimation process from the last data set point. For this we need to use backward-differences and backward difference table. Let us first define backward differences and generate backward difference table, say for the data set $(x_i, f_i), i = 0, 1, 2, 3, 4$.

First order backward difference ∇f_i is defined as:

$$\nabla f_i = f_i - f_{i-1}$$

Second order backward difference $\nabla^2 f_i$ is defined as:

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$$

In general, the k^{th} order backward difference is defined as

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1}$$

In this case the reference point is x_n and therefore we can derive the Newton-Gregory backward difference interpolation polynomial as:

$$P_n(x) = P_n(s) = \sum_{k=0}^{n} \frac{\nabla^k f_n}{k!h^k} \prod_{i=0}^{k-1}(s+i)h$$
$$= \sum_{k=0}^{n} \frac{\nabla^k f_0}{k!h^k} [s(s+1)\ldots\ldots(s+k-1)]h^k \tag{19}$$

Where $s = \dfrac{x - x_n}{h}$ For constructing as given in $Eqn.(18)$ it will be easier if we first generate backward-difference table. The backward difference table for the data $(x_i, f_i), i = 0, 1, 2, 3, 4.$

i	x_i	$f[x_i]$	$\nabla f[x_i]$	$\nabla^2 f[x_i]$	$\nabla^3 f[x_i]$	$\nabla^3 f[x_i]$
0	x_0	$f[x_0]$				
			$\nabla f[x_1]$			
1	x_1	$f[x_1]$		$\nabla^2 f[x_2]$		
			$\nabla f[x_2]$		$\nabla^3 f[x_3]$	
2	x_2	$f[x_2]$		$\nabla^2 f[x_3]$		$\nabla^4 f[x_4]$
			$\nabla f[x_3]$		$\nabla^3 f[x_4]$	
3	x_3	$f[x_3]$		$\nabla^2 f[x_4]$		
			$\nabla f[x_4]$			
4	x_4	$f[x_4]$				

Example 2.6.1. Given the following data, estimate using Newton-Gregory forward difference interpolation polynomial:

x_i	1	3	5	7	9
$y_i = f(x_i)$	0	1.0986	1.6094	1.9459	2.1972

Soln. Here we have five data points i.e $i = 0, 1, 2, 3, 4$. Let us first generate the forward difference table.

i	x_i	$f[x_i]$	$\Delta f[x_i]$	$\Delta^2 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^3 f[x_i]$
0	1	0				
			1.0986			
1	3	1.0986		−0.5878		
			0.5108		0.4135	
2	5	1.6094		−0.1743		−0.3244
			0.3365		0.0891	
3	7	1.9459		−0.0852		
			0.2513			
4	9	2.1972				

$$h = 2, \quad x = 1.83, \quad x_0 = 1.0, \quad s = \frac{x - x_0}{h} = 0.415$$

Newton Gregory forward difference interpolation polynomial is given by:

$$P_1(0.415) = 0 + (0.415)(1.0986) = 0.455919$$

$$P_2(0.415) = 0.455919 + \frac{0.415(0.415-1)}{2}(-0.5878) = 0.455919 + 0.071352 = 0.527271$$

$$P_4(0.415) = 0.527271 + \frac{0.415(0.415-1)(0.415.2)}{6}(0.4135) + \frac{0.415(0.415-1)(0.415-2)(0.415-3)}{24}(-0.3244)$$

$$= 0.554157 + 0.013445 = 0.567602$$

Example 2.6.2. Given the following data estimate $f(4.12)$ using Newton-Gregory forward difference interpolation polynomial:

x_i	0	1	2	3	4	5
$y_i = f(x_i)$	1	2	4	8	16	32

Soln. Let us first generate the Newton-Gregory forward difference table:

i	x_i	$f[x_i]$	$\Delta f[x_i]$	$\Delta^2 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^4 f[x_i]$	$\Delta^5 f[x_i]$
0	0	1					
			1				
1	1	2		1			
			2		1		
2	2	4		2		1	
			4		2		1
3	3	8		4		2	
			8		4		
4	4	16		8			
			16				
5	5	32					

$P_n(4.12) = 1 + (4.12)1 + \frac{4.12(4.12-1)}{2}1 + \frac{4.12(4.12-1)(4.12-2)}{6}1 + \frac{4.12(4.12-1)(4.12-2)(4.12-3)}{24}1$
$+\frac{4.12(4.12-1)(4.12-2)(4.12-3)(4.12-4)}{120}1$
$= 1 + 4.12 + 6.4272 + 4.5419 + 1.2717 + 0.0305 = 17.3913$

Example 2.6.3. Given the following data estimate $f(4.12)$ using Newton-Gregory backward difference interpolation polynomial:

x_i	0	1	2	3	4	5
$y_i = f(x_i)$	1	2	4	8	16	32

Soln. Here

$$x_n = 5, \quad x = 4.12, \qquad h = 1$$

$$\therefore s = \frac{x - x_n}{h} = \frac{4.12 - 5}{1} = -0.88$$

Newton Backward Difference polynomial $P_5(x)$ is given by

$P_n(4.12) = 32 + (-0.88)16 + \frac{(-0.88)(-0.88+1)}{2}8 + \frac{(-0.88)(-0.88+1)(-0.88+2)}{6}4 +$
$\frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)}{24}2 + \frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)(-0.88+4)}{120}1$

$= 32 - 14.08 - 0.4224 - 0.07885 - 0.0209 - 0.0065 = 17.39135$

i	x_i	$f[x_i]$	$\nabla f[x_i]$	$\nabla^2 f[x_i]$	$\nabla^3 f[x_i]$	$\nabla^4 f[x_i]$	$\nabla^5 f[x_i]$
0	0	1					
			1				
1	1	2		1			
			2		1		
2	2	4		2		1	
			4		2		1
3	3	8		4		2	
			8		4		
4	4	16		8			
			16				
5	5	32					

Now one may note from the above problem it is definitely advantageous of use backward difference approach here, as in exactly the same number of steps we are relatively more close to the approximate solution.

Practice Problems

Problem 2.7. 1. Compute $f(1.135)$ using suitable formula from the following table:

x_i	1.140	1.145	1.150	1.155	1.160	1.165
$f(x)$	0.13103	0.135410	0.13976	0.14410	1.14842	0.15272

Ans: 6.65

2. Find the equation of the cubic curve that passes through the points $(0, 5), (1, 10), (2, -9), (3, 4)$ and $(4, 35)$
Ans: $x^3 - 5x^2 + 2x - 3$

Topic No. 3

Numerical differentiation

Learning Objectives At the end of this session students should be able to

1. apply Newton forward differentiation.
2. apply Newton backward differentiation.

THEORITICAL PART

In the case of differentiation, we first write the interpolating formula on the interval (x_0, x_n). and the differentiate the polynomial term by term to get an approximated polynomial to the derivative of the function. When the tabular points are equidistant, one uses either the Newton's Forward/ Backward Formula otherwise Lagrange's formula is used. Newton's Forward/ Backward formula is used depending upon the location of the point at which the derivative is to be computed. We illustrate the process by taking Newton's Forward formula

Recall, that the Newton's forward interpolating polynomial is given by

$$\begin{aligned} f(x) = f(x_0 + hu) \approx y_0 \quad &+ \quad \Delta y_0 u + \frac{\Delta^2 y_0}{2!}(u(u-1)) + \cdots + \frac{\Delta^k y_0}{k!}\{u(u-1)\cdots(u-k+1)\} \\ &+ \quad \cdots + \frac{\Delta^n y_0}{n!}\{u(u-1)...(u-n+1)\}. \end{aligned} \tag{20}$$

Differentiating (1), we get the approximate value of the first derivative at x as

$$\begin{aligned} \frac{df}{dx} = \frac{1}{h}\frac{df}{du} \quad &\approx \quad \frac{1}{h}\left[\Delta y_0 + \frac{\Delta^2 y_0}{2!}(2u-1) + \frac{\Delta^3 y_0}{3!}(3u^2 - 6u + 2) + \cdots \right. \\ &+ \quad \left. \frac{\Delta^n y_0}{n!}\left(nu^{n-1} - \frac{n(n-1)^2}{2}u^{n-2} + \cdots + (-1)^{(n-1)}(n-1)!\right)\right]. \end{aligned} \tag{21}$$

where, $u = \dfrac{x - x_0}{h}$. Thus, an approximation to the value of first derivative at $x = x_0$ i.e. $u = 0$ is obtained as :

$$\left.\frac{df}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \cdots + (-1)^{(n-1)}\frac{\Delta^n y_0}{n}\right].$$

Now higher derivatives can be found by successively differentiating the interpolating polynomials. Thus e.g. using (2), we get the second derivative at $x = x_0$ as

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{2 \times 11}{4!}\Delta^4 y_0 - \cdots\right].$$

Example 3.0.1. Compute from following table the value of the first and second derivative of $y = f(x)$ at $x = 1.5$ from the data.

x_i	1.5	2.0	2.5	3.0	3.5	4.0
$y_i = f(x_i)$	3.375	7.0	13.625	24.0	38.875	59.0

Soln.

x_i	$f[x_i]$	$\Delta f[x_i]$	$\Delta^2 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^4 f[x_i]$
1.5	3.375					
		3.625				
2	7		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4	59					

$$f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2} \times 3 + \frac{1}{3} \times 0.75\right] = 4.75$$

$$f''(1.5) = \frac{1}{0.5^2}\left[3.625 - 0.75\right] = 9$$

Example 3.0.2. Find the values of $\sin 18°$ and $\sin 45°$ from the following table using numerical differentiation based on Newton's forward interpolation formula

x_i	0	10	20	30	40
$\cos(x°)$	1	0.9848	0.9397	0.8660	0.7660

Soln.

x_i	$f[x_i]$	$\Delta f[x_i]$	$\Delta^2 f[x_i]$	$\Delta^3 f[x_i]$	$\Delta^3 f[x_i]$
0	1				
		-0.0152			
10	0.9848		-0.0299		
		-0.0451		0.0013	
20	0.9397		-0.0286		0.001
		-0.0737		0.0023	
30	0.866		-0.0263		

-1

40 0.766

$$\frac{df}{dx} = \frac{1}{h}\frac{df}{du} \approx \frac{1}{h}\left[\Delta y_0 + \frac{\Delta^2 y_0}{2!}(2u-1) + \frac{\Delta^3 y_0}{3!}(3u^2-6u+2) + \frac{\Delta^4 y_0}{4!}(4u^3+18u^2+105u^2+100u+24)\right]$$

$$\text{For } x = 18^\circ u = \frac{18-0}{10} = 1.8 \quad \text{For } x = 45^\circ u = \frac{45-0}{10} = 4.5$$

$$\cos(18^\circ) = -\frac{18}{\pi}\left[-0.0152 + 1.3\times(-0.0299) + \frac{1}{6}\times 0.92\times 0.0013 + \frac{1}{24}\times(13.92)\times 0.001\right] = 0.3120$$

$$\cos(45^\circ) = -\frac{18}{\pi}\left[-0.0152 - 4\times(-0.0299) + \frac{1}{6}\times(35.75)\times(0.0013) + \frac{1}{24}\times 93\times(0.001)\right] = 0.7058$$

Practice Problems

Problem 3.1. 1. Find the values of $\frac{1}{x}$ and $\frac{1}{x^2}$ at $x = 9$ from the following data of values of $y = \ln x$

x_i	1	2	3	4	5	6	7	8	9
y	0	0.6931	1.0986	1.3863	1.6094	1.79182	1.9459	2.0794	2.1972

Ans: 0.11, 0.0123

Topic No. 4

Numerical Integration

Learning Objectives At the end of this session students should be able to

1. apply Trapezoidal rule.
2. apply Simpsons 1/3 and 3/8 rule.
3. apply Weddle rule.

THEORITICAL PART

4.1 General Quadrature Formula

Let $f(x_k) = y_k$ be the nodal value at the tabular point x_k for $k = 0, 1, \cdots, x_n$, where $x_0 = a$ and $x_n = x_0 + nh = b$. Now, a general quadrature formula is obtained by replacing the integrand by Newton's forward difference interpolating polynomial. Thus, we get,

$$\int_a^b f(x)dx = \int_a^b \Bigg[y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \frac{\Delta^4 y_0}{4!h^4}(x - x_0)(x - x_1)(x - x_2)(x - x_3) + \cdots\Bigg] dx \tag{22}$$

This on using the transformation $x = x_0 + hu$ gives:

$$\int_a^b f(x)dx = \int_a^b \Bigg[y_0 + u\Delta y_0 + \frac{\Delta^2 y_0}{2!}u(u - 1) + \frac{\Delta^3 y_0}{3!}u(u - 1)(u - 2) + \frac{\Delta^4 y_0}{4!}u(u - 1)(u - 2)(u - 3) + \cdots\Bigg] du \tag{23}$$

which on term by term integration gives,

$$\int_a^b f(x)dx = \Bigg[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \frac{\Delta^3 y_0}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right) + \frac{\Delta^4 y_0}{4!}\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right) + \cdots\Bigg] \tag{24}$$

For $n = 1$, i.e., when linear interpolating polynomial is used then, we have

$$\int_a^b f(x)dx = h\left[y_0 + \frac{\Delta y_0}{2}\right] = \frac{h}{2}\left[y_0 + y_1\right] \tag{25}$$

Similarly, using interpolating polynomial of degree 2 (i.e. $n = 2$), we obtain,

$$\begin{aligned}\int_a^b f(x)dx &= h\left[2y_0 + 2\Delta y_0 + \left(\tfrac{8}{3} - \tfrac{4}{2}\right)\tfrac{\Delta^2 y_0}{2}\right] \\ &= 2h\left[y_0 + (y_1 - y_0) + \tfrac{1}{3} \times \tfrac{y_2 - 2y_1 + y_0}{2}\right] \\ &= \tfrac{h}{3}\left[y_0 + 4y_1 + y_2\right] \end{aligned} \tag{26}$$

In the above we have replaced the integrand by an interpolating polynomial over the whole interval $[a, b]$ and then integrated it term by term. However, this process is not very useful. More useful Numerical integral formulae are obtained by dividing the interval $[a, b]$ in n sub-intervals $[x_k, x_{k+1}]$, where, $x_k = x_0 + kh$ for $k = 0, 1, \cdots, n$ with $x_0 = a, x_n = x_0 + nh = b$.

4.2 Trapezoidal Rule

Here, the integral is computed on each of the sub-intervals by using linear interpolating formula, i.e. for $n = 1$ and then summing them up to obtain the desired integral. Note that

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{k+1}}^{x_k} f(x)dx + \cdots + \int_{x_n}^{x_{n-1}} f(x)dx$$

Now using the formula (4) for $n = 1$ on the interval $[x_k, x_{k+1}]$, we get,

$$\int_{x_k}^{x_{k+1}} f(x)dx = \frac{h}{2}\left[y_k + y_{k+1}\right].$$

Thus, we have,

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{2}\left[y_0 + y_1\right] + \cdots \frac{h}{2}\left[y_{n-2} + y_{n-1}\right] + \frac{h}{2}\left[y_{n-1} + y_n\right] \\ &= \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_k + \cdots + 2y_{n-1} + y_n\right] \\ &= h\left[\frac{y_0 + y_n}{2} + \sum_{i=1}^{n-1} y_i\right]\end{aligned}$$

This is called **Trapezoidal Rule** It is a simple quadrature formula, but is not very accurate.

Remark: An estimate for the error E_1 in numerical integration using the Trapezoidal rule is given by

$$E_1 = -\frac{b-a}{12}\overline{\Delta^2 y},$$

where $\overline{\Delta^2 y}$ is the average value of the second forward differences. Recall that in the case of linear function, the second forward differences is zero, hence, the Trapezoidal rule gives exact value of the integral if the integrand is a linear function.

4.3 Simpson's Rule

If we are given odd number of tabular points,i.e. n is even, then we can divide the given integral of integration in even number of sub-intervals $[x_{2k}, x_{2k+2}]$. Note that for each of these sub-intervals, we have the three tabular points $x_{2k}, x_{2k+1}, x_{2k+2}$ and so the integrand is replaced with a quadratic interpolating polynomial. Thus using the formula (3), we get,

$$\int_{x_{2k}}^{x_{2k+2}} f(x)dx = \frac{h}{3}\left[y_{2k} + 4y_{2k+1} + y_{2k+2}\right].$$

In view of this, we have

$$\begin{aligned}
\int_a^b f(x)dx &= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{2k}}^{x_{2k+2}} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx \\
&= \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right] \\
&= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right] \\
&= \frac{h}{3}\left[(y_0 + y_n) + 4 \times (y_1 + y_3 + \cdots + y_{2k+1} + \cdots + y_{n-1})\right. \\
&\qquad \left. +2 \times (y_2 + y_4 + \cdots + y_{2k} + \cdots + y_{n-2})\right] \\
&= \frac{h}{3}\left[(y_0 + y_n) + 4 \times \left(\sum_{i=1,\ i\ is\ odd}^{n-1} y_i\right) + 2 \times \left(\sum_{i=2,\ i\ is\ even}^{n-2} y_i\right)\right] \qquad (27)
\end{aligned}$$

This is known as **Simpson's rule**.

Remark: An estimate for the error E_2 in numerical integration using the Simpson's rule is given by

$$E_2 = -\frac{b-a}{180}\overline{\Delta^4 y},$$

where $\overline{\Delta^4 y}$ is the average value of the forth forward differences

Example 4.3.1. Using Trapezoidal rule compute the integral $\int_0^1 e^{x^2} dx$, where the table for the values of $y = e^{x^2}$ is given below:

x_i	0.0	0.1	0.2	0.3	0.4	0.5
y	1.00000	1.01005	1.04081	1.09417	1.17351	1.28402
x_i	0.6	0.7	0.8	0.9	1.0	
y	1.43332	1.63231	1.89648	2.2479	2.71828	

Soln. Here, $h = 0.1,\ n = 10,$

$$\frac{y_0 + y_{10}}{2} = \frac{1.0 + 2.71828}{2} = 1.85914,$$

and

$$\sum_{i=1}^{9} y_i = 12.81257.$$

Thus,

$$\int_0^1 e^{x^2} dx = 0.1 \times [1.85914 + 12.81257] = 1.467171$$

4.4 Simpson's 3/8 Rule

If the number of sub-interval is multiple of three then

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right].$$

$$\therefore \int_a^b f(x)dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3 \times (y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2 \times (y_2 + y_6 + y_9 \cdots + y_{n-3})\Big]$$

This is known as **Simpson's 3/8 rule**.

4.5 Weddle's Rule

If the number of sub-interval is multiple of six then

$$\int_{x_0}^{x_6} f(x)dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right].$$

This is known as **Weddle's rule**.

Practice Problems

Problem 4.6. 1. Solve the following using (i) Trapezoidal rule (ii) Simpson 1/3 rule (iii) Simpson 3/8 rule (iv) Weddle rule.

(a) $\int_0^1 x^3 dx$ taking $n = 6$ *Ans:*

(b) $\int_{1.2}^{1.6} (x + \frac{1}{x})dx$ taking $n = 6$ *Ans:*

(c) $\int_0^{\pi/2} e^{\sin x} dx$ taking $n = 6$ *Ans:*

(d) $\int_{1.2}^{1.6} (4x - 3x^2)dx$ taking $n = 6$ *Ans:*

(e) $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ taking $n = 6$ *Ans:*

(f) $\int_0^1 \frac{dx}{1 + x^2}$ taking $n = 6$ *Ans:*

(g) $\int_0^{\pi/2} \sqrt{1 - 0.162\sin^2 x}\, dx$ taking $n = 6$ *Ans:*

(h) $\int_1^{2.2} \ln(x)dx$ taking $n = 8$ *Ans:*

Topic No. 5

Ordinary differential equation

Learning Objectives At the end of this session students should be able to

1. solve Initial Value Problems by Euler method and Eular modified method
2. solve Initial Value Problems by Runge-Kutta Method of order 2 and 4.

THEORITICAL PART

This lecture attempts to describe a few methods for finding approximate values of a solution of a initial value problems. By no means, it is not an exhaustive study but can be looked upon as an introduction to numerical methods. Again, we stress that no attempt is made towards a deeper analysis. A question may arise: why one needs numerical methods for differential equations. Probably because, differential equations play an important role in many problems of engineering and science. This is so because the differential equations arise in mathematical modelling of many physical problems. The use of the numerical methods have become vital in the absence of explicit solutions. Normally, numerical methods have two major roles:

1. the amicability of the method for easy implementation on a computer;
2. the method allows us to deal with the analysis of error estimates.

In this chapter, we do not enter into the aspect of error analysis. For the present, we deal with some of the numerical methods to find approximate value of the solution of initial value problems (IVP's) on finite intervals. We also mention that no effort is made to study the boundary value problems. Let us consider an initial value problem

$$y' = f(x, y),\ a \leq x \leq b,\ y(a) = y_0 \tag{28}$$

where $f : [a,\ b] \longrightarrow \mathbb{R},\ a, b$ and y_0 are prescribed real numbers and $|b - a| < \infty$.

5.1 Euler's Method

For a small step size h , the derivative $y'(h)$ is close enough to the ratio $\dfrac{y(x+h) - y(x)}{h}$. In the Euler's method, such an approximation is attempted. To recall, we consider the problem (1). Let $h = \dfrac{b-a}{n}$ be the step size and let $x_i = a + ih, 0 \leq i \leq n$ with $x_n = b$. Let y_k be the approximate value of y at $x_k,\ k = 1, 2 \ldots, n$. We define

$$y_{k+1} = y_k + hf(x_k, y_k),\ k = 0, 1, 2, \ldots, n-1 \tag{29}$$

The method of determination of y_k by (2) is called the **Eular's Method**. For convenience, the value $f(x_k, y_k)$ is denoted by f_k , for $k = 0, 1, 2, \dots, n$.

Remark:

1. Euler's method is an one-step method.

2. The Euler's method has a few motivations.

   (a) The derivative y' at $x = x_i$ can be approximated by $\dfrac{y(x_{i+1}) - y(x_i)}{h}$ if h is sufficiently small. Using such an approximation in (1), we have

   $$f(x_i, y(x_i)) = y'(x_i) \cong \frac{y(x_{i+1}) - y(x_i)}{h}.$$

   (b) We can also look at (2) from the following point of view. The integration of (1) yields

   $$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y(x))\, dx.$$

   The integral on the right hand side is approximated, for sufficiently small value of $h > 0$, by

   $$\int_{x_i}^{x_{i+1}} f(s, y(x))\, dx \cong f(x_i, y(x_i))h.$$

   (c) Moreover, if y is differentiable sufficient number of times, we can also arrive at (2) by considering the Taylor's expansion

   $$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2} f(x_i, y(x_i)) + \cdots$$

   and neglecting terms that contain powers of h that are greater than or equal to 2.

3. A great disadvantage of the method lies in the fact that if h is not small enough then the method yields erroneous result; on the other hand, if h is taken too small enough then the method becomes very slow.

Example 5.1.1. Use Euler's algorithm to find an approximate value of $y(1)$, where y is the solution of the IVP

$$y' = y^2, \;\; y(0) = 1, \; 0 \le x \le 0.5$$

with step sizes (i) 0.1 and (ii) 0.05. The solution of the IVP is $y(x) = \dfrac{1}{1-x}$. Calculate the error at each step and tabulate the results.

Soln.

$$f(x, y) = y^2, \; a = 0, \; b = 0.5, \;\; \text{and } y_0 = 1.$$

The Euler's algorithm now reads as

$$y_{k+1} = y_k + hy_k^2, \; k = 0, 1, 2, \dots \text{ and } y_0 = 1.$$

It is left as an exercise to verify that $y(x) = \dfrac{1}{1-x}$ is a solution of the given IVP. So, the absolute value of the error at the j^{th} step is

$$\text{Absolute Error} = |y(x_j) - y_j| = \left|\frac{1}{1-x_j} - y_j\right|.$$

Initial x	Initial y	Step size h	Approx y	Exact y	Error
	1.00000	0.10000			
	1.00000	0.10000	1.10000	1.11111	
0.20000	1.10000	0.10000	1.22100	1.25000	0.02900
0.30000	1.22100	0.10000	1.37008	1.42857	0.05849
0.40000	1.37008	0.10000	1.55780	1.66667	0.10887
0.50000	1.55780	0.10000	1.80047	2.00000	0.19953

5.2 Modified Euler's Method

To remove the drawback to some extent, we shall discuss modified Eular's Method starting with the initial value y_0 an approximate value for y_1. From eular's method the approximate value of y_1 is as:

$$y_1^{(0)} = y_0 + hf(x_0, y_0) \tag{30}$$

Then to get the second approximation for y_1 we replace $f(x_0, y_0)$ in(3) by average value of $f(x_0, y_0)$ and $f(x_1, y_1^{(0)})$. Thus the second approximation for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \tag{31}$$

similarly, third approximation for y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

Thus, in general

$$y_n^{(k)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(k-1)})\right] \quad k = 1, 2, 3, \cdots \tag{32}$$

is used to approximate y_n.

Example 5.2.1. Use Modified Euler's methods to find an approximate value of $y(1.2)$, where y is the solution of the IVP

$$y' + \frac{y}{x} = \frac{1}{x^2}, \quad y(1) = 1,$$

correct upto 4 decimal place.

Soln.

$$f(x,y) = \frac{1}{x^2} - \frac{y}{x},\ x_0 = 1,\ y_0 = 1$$

Let $h = 0.1$ so that $x_1 = 1 + 0.1 = 1.1$

$$\therefore y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1 = 1.1$$

now we modified the y_1 as follows:

iteration (k)	$y_1^k = 1 + 0.05\left(\frac{1}{1.1^2} - \frac{y_1^{(k-1)}}{1.1}\right)$
1	0.99587
2	0.99606
3	0.99607

Hence $y(1.1) = 0.9961$

$$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.9961 + 0.1(-0.079) = 0.98819$$

iteration (k)	$y_2^k = 0.9961 + 0.05\left(-0.079 + \frac{1}{1.2^2} - \frac{y_1^{(k-1)}}{1.2}\right)$
1	0.98569
2	0.98580
3	0.985797

Hence $y(1.2) = 0.985797$

Practice Problems

Problem 5.3. 1. Find the solution of the differential equation

$$\frac{dy}{dx} = x^2 - y,\ y(0) = 1$$

for $x = 0.3$ taking $h = 0.1$ and using Eular's method. Compare the result with the exact solution.

Ans: $0.7492, 0.0153$

2. Using Euler's method, find an approximate value of y at 0.5 given that

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$

Ans: 1.72

3. Solve the equation

$$5x\frac{dy}{dx} + y^2 - 2 = 0,\ y(4) = 1$$

for $y(4.1)$, taking $h = 0.1$ and using modified Euler's method. *Ans:* 1.005

5.4 Runga- Kutta Method

Runga- Kutta method is a more general and improvised method than that of the Euler's method. It uses, as we shall see, Taylor's expansion of a smooth function. Before we proceed further, the following questions may arise in our mind, which has not found place in our discussion.

(a) How does one choose the starting values, sometimes called starters that are required to implement an algorithm ?

(b) Is it desirable to change the step size (or the length of the interval) h during the computation if the error estimates demands a change in h?

For the present, the discussion about question (b) is not taken up. We try to look more on question (a) in the ensuing discussion. There are many self-starter methods, like the Euler method which uses the initial condition. But these methods, are normally, not very efficient since the error bounds may not be "good enough". The local error (neglecting the rounding-off error) is $O(h^2)$ in the Euler's Algorithm. This shows that the smaller the value of h, better the approximations are, But a smaller values of the step size h increases the volume of computations. Moreover the error of order $O(h)$ (h is the step size) may not be sufficiently accurate for many problems. So we look into a few methods where the error is of higher order. They are Runge-Kutta (in short R-K) methods. Let us analyze how the algorithm is reached before we actually state it. We consider the IVP

$$y' = f(x, y), \quad y(a) = y_0 \quad x\epsilon[a, b]$$

. Define $x_k = x_0 + kh$ for $k = 0, 1, 2 \cdots n$ with $x_0 = a$ and $x_n = b$. We now assume that both y and f are "smooth" functions (thereby we mean that the derivatives, could be partial also exists and are continuous upto certain "desired" order). Using Taylor's series, we now have

$$y(x_{k+1}) = y(x_k) + hy'(x_k) + \frac{h^2}{2}y''|x_k| + \frac{h^3}{6}y''|x_k| + \cdots \tag{33}$$

(y_{k+1} denotes the approximate value of y at x_{k+1}, to be defined shortly.) Consider, for $k = 0, 1, 2 \cdots n-1$

$$y_{k+1} = y_k + pk_1 + qk_2 + \cdots \tag{34}$$

where

$$k_1 = hf(x_k, y_k) \tag{35}$$

$$k_2 = hf(x_k + \alpha h, y_k + \beta k_1) \tag{36}$$

where p, q, and β are constants. When , (34) reduces to the Euler Algorithm. We choose p, q, and β so that the local truncation error is). From the definition of k_2 we have

$$\frac{k_2}{h} = f(x_k, y_k) + \alpha hf_x + \beta k_1 f_y + \frac{\alpha^2 h^2}{2} f_{xx} + \alpha\beta hk_1 f_{xy} + \frac{\beta^2 k^2}{2} f_{yy} + O(h^3)$$

where f_x, f_y, denotes the partial derivatives of f w.r.t. x, y, respectively. Substituting these values in (34), we have

$$y_{k+1} = y_k + h(p+q)f + qh^2(\alpha f_x + \beta f f_y) + qh^3\left(\frac{\alpha^2}{2}.f_{xx}\alpha\beta f f_{xy} + \frac{p^2}{2}f^2 f_{yy}\right) + O(h^4) \tag{37}$$

A comparison of (33) and (36), leads to the choice

$$p + q = 1 \tag{38}$$

$$q(q(\alpha f_x + \beta f_y) = \frac{1}{2}y'' \tag{39}$$

in order that the powers of h upto match (in some sense) in the approximate values of y_{k+i}. Here we note that

$$y'' = f_x + f_y f$$

Now we choose α, β,p and q so that (38) and(39) are satisfied. One of the simplest solution is $p = q = \frac{1}{2}$ and $\alpha = \beta = 1$ Thus, we are lead to define y_{k+i} by

$$y_{k+1} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_k + h, y_k + hf(x_k))\right] \tag{40}$$

Evaluation of y_{k+1} by (40) is called the Runge-Kutta method of order 2 (R-K method of order 2) .

A few things are worthwhile to be noted in the above discussion. Firstly, we need the existence of partial derivatives of f upto third order for R-K method of order 2. For higher order methods, we need f to be more smooth. Secondly we note that the local truncation error(in R-K method of order 2) is of the order $O(h^3)$. Again, we remind the readers here that the round off error in the case of implementation has not been considered. Also in (40), the partial derivatives of f do not appear. In short, we are likely to get a better accuracy in Runge-Kutta method of order 2 in comparison with the Euler's method. Formally, we state the Runge-Kutta method of order 2.

Example 5.4.1. Use the Runge-Kutta method of order 2 to find the approximate $y(0.2)$ and $y(0.4)$ of the IVP.

$$yy' = y^2 - x, \ \ y(0) = 2$$

Soln.

$$f(x, y) = y' = \frac{y^2 - x}{y}, \ \ x_0 = 0, y_0 = 2$$

consider $h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2 \times \frac{2^2 - 0}{2} = 0.4$$

$$k_2 = 0.2 \times \frac{2.4^2 - 0.2}{2.4} = 0.46333$$

$$\text{Thus,} \quad y(0.2) = y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 2.43166$$

$$k_1 = hf(x_1, y_1) = 0.2 \times \frac{(0.432)^2 - 0.2}{0.432} = -0.00633$$
$$k_2 = 0.2 \times \frac{2^2 - 0}{2} = -0.10302$$
$$\text{Thus,} \quad y(0.4) = y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 2.37698$$

Practice Problems

Problem 5.5. 1. Find the solution of the differential equation

$$\frac{dy}{dx} = x - y^2, \; y(0) = 1$$

for $x = 0.2$ taking $h = 0.1$ by using fourth order RK method. *Ans:* 0.851

2. Find the solution of the differential equation

$$\frac{dy}{dx} = -xy, \; y(0) = 1$$

for $x = 0.2$ taking $h = 0.1$ by using fourth order RK method. *Ans:* 0.9802

3. Find the solution of the differential equation

$$\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$$

for $x = 0.4$ taking $h = 0.1$ by using fourth order RK method. *Ans:* 0.4228

5.6 Runge-Kutta Method of order 4

There are generalization of R-K Method of order 2 to higher order methods. Without getting into analytical details , we state the R-K method of order 4. It is a widely used algorithm. For the initial value problems (34), set $x_i = a + ih$, $i = i = 0, 1, 2 \cdots n$ and $y_0 = y_0$ for $k = 0, 1, 2, \cdots n-1$, define y_{k+1} by

$$y_{k+1} = y_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_k, y_k) \quad (41)$$
$$k_2 = hf(x_k + \frac{h}{2}, y_k + \frac{k_1}{2}) \quad (42)$$
$$k_3 = hf(x_k + \frac{h}{2}, y_k + \frac{k_2}{2}) \quad (43)$$
$$k_2 = hf(x_k + h, y_k + k_3) \quad (44)$$

Example 5.6.1. Use the Runge-Kutta method of order 4 to find the approximate $y(1.1)$ of the IVP.

$$y' = y^2 + xy, \;\; y(1) = 1$$

Soln.

$$f(x,y) = y' = y^2 + xy, \;\; x_0 = 1, y_0 = 1$$

$$k_1 = hf(x_0, y_0) = 0.1(1^2 + 1 \times 1) = 0.2$$
$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1(1.1^2 + 1.05 \times 1.1) = 0.2365$$
$$k_3 = hf(x_k + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1((1.11825)^2 + 1.05 \times 1.11825) = 0.2425$$
$$k_2 = hf(x_k + h, y_k + k_3) = 0.1((1.2425)^2 + 1.05 \times 1.2425) = 0.2910556$$

$$\text{Thus,} \quad y(1.1) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2415$$

Practice Problems

Problem 5.7. 1. Find the solution of the differential equation

$$\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$$

for $x = 0.2$ by using fourth order RK method. *Ans:* 0.2024

2. Using fourth order RK method, find an approximate value of y at 0.2 taking step size $h = 0.1$ given that

$$\frac{dy}{dx} = x + y, \; y(0) = 1$$

Ans: 1.2205

3. Solve the equation

$$\frac{dy}{dx} = xy + y^2, \; y(0) = 1$$

for $y(0.1)$, $y(0.2)$ $y(0.3)$, taking $h = 0.1$ using fourth order RK method. *Ans:* 0.1168, 1.2689, 1.4856

Topic No. 6

System of linear equation

Learning Objectives At the end of this session students should be able to

1. recall the definition of moments of a random variable.
2. compute the kth moment about any fixed point.
3. recall the definition of central moments and compute kth central moment.
4. state the definition of moment generating functions and compute them.

THEORITICAL PART

6.1 The Elimination Method

Consider an upper-triangular system given by

$$5x_1 + 3x_2 - 2x_3 = -3$$

$$6x_2 + x_3 = -1$$

$$2x_3 = 10$$

It is very easy to obtain its solution. From the last equation, we see that $x_3 = 5$ and substituting in the 2^{nd} equation gives $x_2 = -1$. Finally, substituting these values in the first equation gives $x_1 = 2$. Thus the solution is $x_1 = 2, x_2 = -1, x_3 = 5$.

The first objective of the elimination method is to reduce the matrix of coefficients to an upper-triangular form. Consider this example of three equation:

$$4x_1 - 2x_2 + x_3 = 15$$

$$-3x_1 - x_2 + 4x_3 = 8$$

$$x_1 - x_2 + 3x_3 = 13$$

We first eliminate x_1 from the 2^{nd} and 3^{rd} equation. This is done by performing the calculations as $4R_2 + 3R_1$ and $4R_3 - R_1$ (where R_i stands for the i^{th} row), we get

$$4x_1 - 2x_2 + x_3 = 15$$

$$-10x_2 + 19x_3 = 77$$

$$-2x_2 + 11x_3 = 37$$

We now eliminate x_2 from the third equation; this is done by performing the calculations as $-10R_3 + 2R_2$ to get

$$4x_1 - 2x_2 + x_3 = 15 \tag{45}$$

$$-10x_2 + 19x_3 = 77 \tag{46}$$

$$-72x_3 = -216 \tag{47}$$

This yields solution, by backward substitution, as

$$x_3 = 3, x_2 = -2, x_1 = 2$$

We now present the above problem, solved in exactly the same way, in matrix notation. We write the given system as

$$\begin{bmatrix} 4 & -2 & 1 \\ -3 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 13 \end{bmatrix}$$

and form the augmented matrix

$$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ -1 & 1 & 3 & 13 \end{array}\right]$$

We carry out the elementary row transformations to convert A to upper triangular form. Using the transformation as given above, we get

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -72 & -216 \end{bmatrix}$$

and using backward substitution yields the solution. Note that there exists the possibility that the set of equations has no solution, or that the prior procedure will fail to find it. During the triangularization step, if a zero is encountered on the diagonal, we cannot use that row to eliminate coefficients below that zero element. However, in that case, we can continue by interchanging rows and eventually achieve an upper triangular matrix of coefficients. The real stumbling block is finding a zero on the diagonal after we have triangularized. If that occurs, it means that the determinat is zero and there is no solution. Let us now state what we mean by elementary row operations, that we have used to solve the above system.

There are three of these operations:

1. We may multiply any row of the augmented matrix by a constant.
2. We can add a multiple of one row to a multiple of any other row.
3. We can inter change the order of any two rows.

It is intuitively obvious that all the three above operations do not change the solution of the system.

Example 6.1.1. Solve the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 3x_3 &= 3 \\ -x_1 - 3x_2 \qquad &= 2 \end{aligned}$$

Soln. Represent the linear system by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right]$$

and carry out the row operations as given below

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1/2 & 1/2 \end{array}\right]$$

Solving We get

$$x_3 = 1, \ \ x_2 = -1, \ \ x_1 = 1$$

Now one may note from the above problem it is definitely advantageous of use backward difference approach here, as in exactly the same number of steps we are relatively more close to the approximate solution.

Practice Problems

Problem 6.2. (1) Solve the following problem by Gauss elimination method

$$(a) \begin{cases} 2x + \ \ y + 3z = 10 \\ 3x + 2y + 3z = 12 \\ \ \ x + 4y + 9z = 16 \end{cases} \quad (b) \begin{cases} 2x + \ \ 2y + \ \ z = 10 \\ 3x + \ \ 2y + 2z = 8 \\ 5x + 10y - 8z = 10 \end{cases} \quad (c) \begin{cases} \ \ x + 4y + 3z = -5 \\ 3x + 2y + \ \ z = -12 \\ 3x - \ \ y - \ \ z = 10 \end{cases}$$

6.3 Jacobis Method

Carl Gustav Jacob Jacobi (1804-1851) gave an indirect method for finding the solution of a system of linear equations, which is based on the successive better approximations of the values of the unknowns, using an iterative procedure. The sufficient condition for the convergence of Gauss Jacobi method to solve $Ax = b$ is that the coefficient matrix A is strictly diagonally row dominant, that is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \text{ then,}$$

$$a_{ii} > \sum_{j=1 j\neq i}^{n} |a_{ij}|$$

It should be noted that this method makes two assumptions. First, the system of linear equations to be solved, must have a unique solution and second, there should not be any zeros on the main diagonal of the coefficient matrix A. In case, there exist zeros on its main diagonal, then rows must be interchanged to obtain a coefficient matrix that does not have zero entries on the main diagonal. Consider a system of n linear equations in n unknowns, which are strictly diagonally row dominant, as follows:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

Since the system is strictly diagonally row dominant, $a_{ii} \neq 0$

Therefore, the system of equations is rewritten as

$$x_1 = \frac{b_1}{a_{11}} - 0\cdot x_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 + \cdots - \frac{a_{1n}}{a_{11}}x_n$$
$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - 0\cdot x_2 - \frac{a_{23}}{a_{22}}x_3 + \cdots - \frac{a_{2n}}{a_{22}}x_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 + \cdots - 0\cdot x_n$$

We then consider an arbitrary initial guess of the solution as $(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$. which are row substituted to the right hand side of the rewritten equations to obtain the first approximation as

$$x_1^{(1)} = \frac{b_1}{a_{11}} - 0\cdot x_1^{(0)} - \frac{a_{12}}{a_{11}}x_2^{(0)} - \frac{a_{13}}{a_{11}}x_3^{(0)} + \cdots - \frac{a_{1n}}{a_{11}}x_n^{(0)}$$
$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(0)} - 0\cdot x_2^{(0)} - \frac{a_{23}}{a_{22}}x_3^{(0)} + \cdots - \frac{a_{2n}}{a_{22}}x_n^{(0)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n^{(1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(0)} - \frac{a_{n2}}{a_{nn}}x_2^{(0)} - \frac{a_{n3}}{a_{nn}}x_3^{(0)} + \cdots - 0\cdot x_n^{(0)}$$

This process is repeated by substituting the first approximate solution $(x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)})$ to the r.h.s of the rewritten equations. By repeated iteration, we get the required solution up to the desired level of the accuracy.

Example 6.3.1. solve the linear system

$$x_1 + x_2 + 4x_3 = 9$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Soln. The given system of equations is not diagonally row dominant as $|a_{22}| < |a_{12}| + |a_{13}|$. Therefore, we re-arrange the system as

$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$x_1 + x_2 + 4x_3 = 9$$

Thus, the system is diagonally row dominant. We now re-write the system as:

$$x_1 = \frac{1}{8}(20 - 3x_2 - 2x_3)$$
$$x_2 = \frac{1}{11}(33 - 4x_1 + x_3)$$
$$x_3 = \frac{1}{4}(9 - x_1 - x_2)$$

Let the initial guess be $x_1^{(0)} = 1, x_2^{(0)} = 1, x_3^{(0)} = 0$ Then, the first approximation to the solution is given by

$$x_1^{(1)} = \frac{1}{8}(20 - 3 \times 1 - 2 \times 0) = 2.875$$
$$x_2^{(1)} = \frac{1}{11}(33 - 4 \times 1 + 0) = 2.636$$
$$x_3^{(1)} = \frac{1}{4}(9 - 1 - 1) = 1.75$$

2nd approx	3rd approx	4th approx	5th approx	6th approx
$x_1^{(2)} = 3.051$	$x_1^{(3)} = 3.075$	$x_1^{(4)} = 2.999$	$x_1^{(5)} = 2.991$	$x_1^{(6)} = 2.997$
$x_2^{(2)} = 2.114$	$x_2^{(3)} = 1.969$	$x_2^{(4)} = 1.969$	$x_2^{(5)} = 1.999$	$x_2^{(6)} = 2.004$
$x_3^{(2)} = 0.872$	$x_3^{(3)} = 0.981$	$x_3^{(4)} = 0.989$	$x_3^{(5)} = 0.989$	$x_3^{(6)} = 1.002$

Therefore,$x_1 = 3.0, x_2 = 2.0, x_3 = 1$ correct to two significant figures.

6.4 Gauss Seidel Method

Gauss Seidel iteration method for solving a system of n-linear equations in n- unknowns is a modified Jacobis method. Therefore, all the conditions that is true for Jacobis method, also holds for Gauss Seidel method. As before, the system of linear equations are rewritten as

$$x_1 = \frac{b_1}{a_{11}} - 0 \cdot x_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 + \cdots - \frac{a_{1n}}{a_{11}}x_n$$
$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - 0 \cdot x_2 - \frac{a_{23}}{a_{22}}x_3 + \cdots - \frac{a_{2n}}{a_{22}}x_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 + \cdots - 0 \cdot x_n$$

If $(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$ be the initial guess of the solution, which is arbitrary, then the first approximation to the solution is obtained as.

$$
\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - 0 \cdot x_1^{(0)} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} + \cdots - \frac{a_{1n}}{a_{11}} x_n^{(0)} \\
x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - 0 \cdot x_2^{(0)} - \frac{a_{23}}{a_{22}} x_3^{(0)} + \cdots - \frac{a_{2n}}{a_{22}} x_n^{(0)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \frac{a_{n2}}{a_{nn}} x_2^{(1)} - \frac{a_{n3}}{a_{nn}} x_3^{(1)} + \cdots - 0 \cdot x_n^{(0)}
\end{aligned}
$$

Please note, while calculating $x_2^{(1)}$ the value of $x_1$is replaced by $x_1^{(1)}$ not by $x_1^{(0)}$. This is the basic difference of Gauss Seidel with Jocobis method The successive iterations are generated by the scheme called iteration formulae of Gauss-Seidel method, which is as follows:

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{b_1}{a_{11}} - 0 \cdot x_1^{(k)} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} + \cdots - \frac{a_{1\,n-1}}{a_{11}} x_{n-1}^{(k)} - \frac{a_{1n}}{a_{11}} x_n^{(k)} \\
x_2^{(k+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - 0 \cdot x_2^{(k)} - \frac{a_{23}}{a_{22}} x_3^{(k)} + \cdots - \frac{a_{2\,n-1}}{a_{22}} x_{n-1}^{(k)} - \frac{a_{2n}}{a_{22}} x_n^{(k)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n^{(k+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(k+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(k+1)} - \frac{a_{n3}}{a_{nn}} x_3^{(k+1)} + \cdots - \frac{a_{n-1\,3}}{a_{nn}} x_{n-1}^{(k+1)} - 0 \cdot x_n^{(k)}
\end{aligned}
$$

Example 6.4.1. solve the linear system Gauss Seidel method

$$
\begin{aligned}
x_1 + x_2 + 4x_3 &= 9 \\
8x_1 - 3x_2 + 2x_3 &= 20 \\
4x_1 + 11x_2 - x_3 &= 33
\end{aligned}
$$

Soln. The given system of equations is not diagonally row dominant as $|a_{22}| < |a_{12}| + |a_{13}|$. Therefore, we re-arrange the system as

$$
\begin{aligned}
8x_1 - 3x_2 + 2x_3 &= 20 \\
4x_1 + 11x_2 - x_3 &= 33 \\
x_1 + x_2 + 4x_3 &= 9
\end{aligned}
$$

Thus, the system is diagonally row dominant. We now re-write the system as:

$$
\begin{aligned}
x_1 &= \frac{1}{8}(20 - 3x_2 - 2x_3) \\
x_2 &= \frac{1}{11}(33 - 4x_1 + x_3) \\
x_3 &= \frac{1}{4}(9 - x_1 - x_2)
\end{aligned}
$$

Let the initial guess be $x_1^{(0)} = 1, x_2^{(0)} = 1, x_3^{(0)} = 0$ Then, the first approximation to the solution is given by

$$x_1^{(1)} = \frac{1}{8}(20 - 3 \times 1 - 2 \times 0) = 2.875$$
$$x_2^{(1)} = \frac{1}{11}(33 - 4 \times 2.875 + 0) = 1.995$$
$$x_3^{(1)} = \frac{1}{4}(9 - 2.875 - 1.995) = 1.043$$

2nd approx	3rd approx	4th approx
$x_1^{(2)} = 2.972$	$x_1^{(3)} = 3.004$	$x_1^{(4)} = 3.00$
$x_2^{(2)} = 2.014$	$x_2^{(3)} = 1.999$	$x_2^{(4)} = 2.00$
$x_3^{(2)} = 1.004$	$x_3^{(3)} = 0.999$	$x_3^{(4)} = 1.00$

Therefore,$x_1 = 3, x_2 = 2, x_3 = 1$ correct to two significant figures.

Practice Problems

Problem 6.5. (1) Solve the following problem by Gauss Seidel and Jecobi method

$$(a) \begin{cases} 6x + 3y + 12z = 36 \\ 8x - 3y + 2z = 20 \\ 4x + 11y - z = 33 \end{cases} \quad (b) \begin{cases} 9x - 3y + 2z = 23 \\ 6x + 3y + 14z = 38 \\ 4x + 12y - z = 35 \end{cases} \quad (c) \begin{cases} 5x + 2y + z = -12 \\ x + 4y + 2z = 15 \\ x + 2y + 5z = 20 \end{cases}$$

Topic No. 7

Partial differential equation

Learning Objectives At the end of this session students should be able to

1. Classified the partial differential equation of first and second order

THEORITICAL PART

A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables.

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = z, \quad z = z(x, y)$
2. $\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = xy.$

7.1 Order of partial differential equation

The order of a partial differential equation is the order of the highest derivative involved in the partial differential equation.

7.2 Degree of partial differential equation

Highest power of the highest order derivative occurring in the given partial differential equation is referred as the degree of the partial differential equation, after making it free from radicals, fractions and transcandal functions as far as derivatives are concerned.

Notations

$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \ \frac{\partial^2 z}{\partial y^2} = t,\ \ \frac{\partial^2 z}{\partial x \partial y} = s.$

7.3 Classification of the first order partial differential equation

1. **Linear partial differential equation:** A partial differential equation of the form
   $P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$ is called the linear PDE of first order.

2. **Semi-linear partial differential equation:** A partial differential equation of the form
   $P(x, y)p + Q(x, y)q = R(x, y,\ z)$ is called the semi-linear PDE.

3. **Quasi-linear partial differential equation:** A partial differential equation of the form
   $P(x, y, z)p + Q(x, y, z)q = R(x, y,\ z)$ s called the quasi-linear PDE.

4. **Non-linear partial differential equation:** A first order partial differential equation which does not come under any of the above types is called the non-linear PDE.

7.4 Classification of the second order partial differential equation

1. **Linear partial differential equation:** If given second order partial differential equation (PDE) is of the form
$[R(x, y)r + S(x, y)s + T(x, y)t] + P(x, y)p + Q(x, y)q + M(x, y)z = N(x, y)$ is called the linear partial differential equation of second order.

2. **Semi-linear partial differential equation:** If the PDE is of the form
$R(x, y)r + S(x, y)s + T(x, y)t + f(x, y, z, p, q) = 0$ is called the semi linear partial differential equation of second order.

3. **Quasi-linear partial differential equation:** If the PDE is of the form
$R(x, y, z, p, q)r + S(x, y, z, p, q)s + T(x, y, z, p, q)t + f(x, y, z, p, q) = 0$ is called the quasi linear partial differential equation of second order.

4. **Non-linear partial differential equation:** A second order partial differential equation which does not come under any of the above types is called the non-linear PDE of second order.

7.5 Classification of the second order semi-linear partial differential equation

Consider the semi-linear partial differential equation

$$R(x, y)r + S(x, y)s + T(x, y)t + f(x, y, z, p, q) = 0 \tag{48}$$

Or

$$Ar + Bs + Ct + f(x, y, z, p, q) = 0 \tag{49}$$

(49) is called

$$Hyperbolic \quad if \quad B^2 - 4AC > 0, \tag{50}$$

$$Elliptic \quad if \quad B^2 - 4AC < 0, \tag{51}$$

$$Parabolic \quad if \quad B^2 - 4AC = 0 \tag{52}$$

Question If $u_{xx} + u_{yy} + u_{zz} + 2x^2u_{xz} = 0$ then the given partial differential equation is

1. Parabolic for $|x| = 1$

2. Hyperbolic for $|x| > 1$

3. Elliptic for $|x| < 1$

4. Elliptic for $|x| > 1$

Ans: 2, 3