$$= \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2}\, dx = \left[\frac{x^6}{32a^2 \times 6}\right]_0^{2a} = \frac{a^4}{3}$$

Type 2. Evaluation of a double integral by changing the order of integration

1. *Change the order of integration and hence evaluate* $\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx.$

Solution $\qquad y = 2\sqrt{ax}$

$\Rightarrow \qquad y^2 = 4ax$

when $x = a$ on $y^2 = 4ax$, $y^2 = 4a^2$

$\Rightarrow \qquad y = \pm\, 2a$

So, on $y = 2\sqrt{ax}$, $y = 2a$ when $x = a$

The integral is over the shaded region.

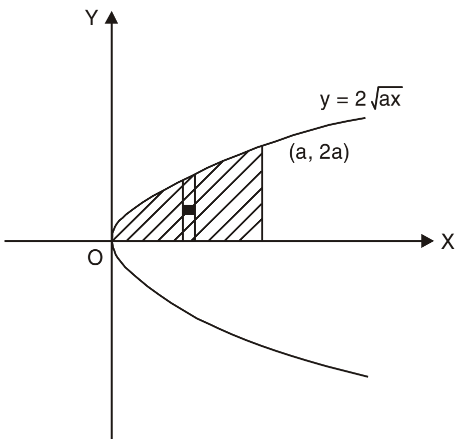

Fig. 3.9

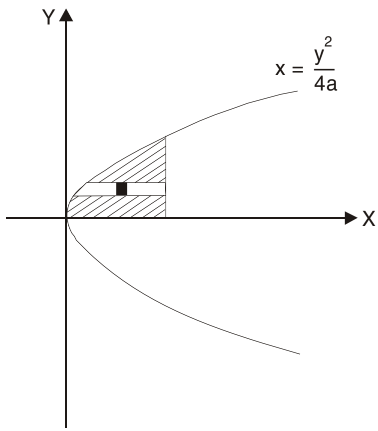

Fig. 3.10

$$\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx = \int_{y=0}^{2a} \int_{x=\frac{y^2}{4a}}^{a} x^2\, dx\, dy \qquad \text{(By changing the order)}$$

$$= \int_0^{2a} \left[\frac{x^3}{3}\right]_{\frac{y^2}{4a}}^{a} dy$$

$$= \int_0^{2a} \left(\frac{a^3}{3} - \frac{y^6}{192\, a^3} \right) dy$$

$$= \left[\frac{a^3}{3}\, y - \frac{y^7}{192 a^3 \times 7} \right]_0^{2a}$$

$$= \frac{2a^4}{3} - \frac{2^7\, a^4}{192 \times 7}$$

$$= a^4 \left(\frac{2}{3} - \frac{2}{21} \right) = \frac{4}{7} a^4.$$

2. *Change the order of integration and hence evaluate* $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$.

Solution $\quad y = \sqrt{2 - x^2}$

$\Rightarrow \quad y^2 = 2 - x^2$

$\Rightarrow \quad x^2 + y^2 = 2$

This circle and $y = x$ meet if $x^2 + x^2 = 2$

$\therefore \quad 2x^2 = 2 \Rightarrow x = 1$

So, (1, 1) is the meeting point.

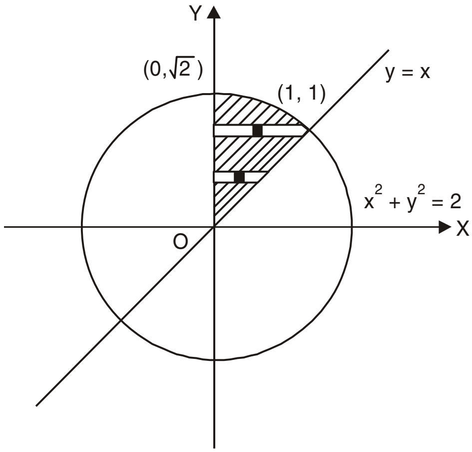

Fig. 3.11

Now $$I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$= \int_{y=0}^{\sqrt{2}} \int_{x=0}^{\phi(y)} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

where $\phi(y) = \begin{cases} y & \text{for } 0 \le y \le 1 \\ \sqrt{2-y^2} & \text{for } 1 \le y \le \sqrt{2} \end{cases}$

(Note that $x = \phi(y)$ is the R.H.S. boundary of the shaded region)

So, the required integral is

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \int_0^1 \left[x^2 + y^2 \right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2} \right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{2}\, y - y \right) dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y \right) dy$$

$$= \left[\left(\sqrt{2}-1\right)\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}\,y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2} + \sqrt{2}\left(\sqrt{2}-1\right) - \left(\frac{2}{2}-\frac{1}{2}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}.$$

3. *Change the order of integration and hence evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$

Solution. The region of integration is the portion of the first quadrant between $y = x$ and the y-axis. So, by changing the order of integration.

$$\int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,[x]_0^y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[-e^{-y}\right]_0^\infty = 1.$$

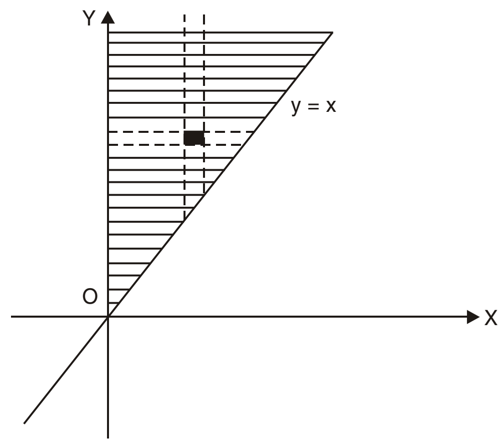

Fig. 3.12

4. *Change the order of integration and hence evaluate* $\int_{y=0}^{3} \int_{x=1}^{4-y} (x+y)\, dx\, dy.$

Solution $\quad x = 4 - y \quad \Rightarrow \quad x + y = 4$

Limits for x are from 1 to $4 - y$

when $\quad x = 1$ on $x + y = 4$

we have $\quad 1 + y = 4 \;\Rightarrow\; y = 3$

So, $\int_0^3 \int_1^{4-y} (x+y)\, dx\, dy = \int_{x=1}^{4} \int_0^{4-x} (x+y)\, dy\, dx$

by changing the order of integration.

$$= \int_1^4 \left[xy + \frac{y^2}{2}\right]_0^{4-x} dx$$

$$= \int_1^4 \left[x\,(4-x) + \frac{(4-x)^2}{2}\right] dx$$

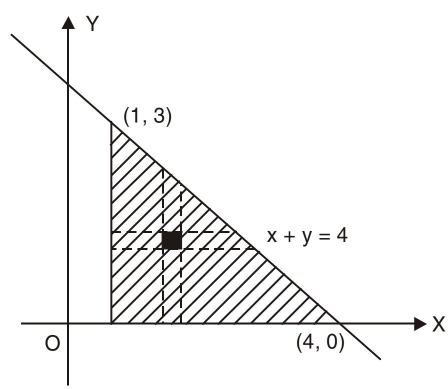

Fig. 3.13

$$= \int_1^4 \left(8 - \frac{1}{2}x^2\right) dx$$

$$= \left[8x - \frac{x^3}{6}\right]_1^4 = \frac{27}{2}.$$

5. *Change the order of integration and hence evaluate* $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy.$

Solution $x = \sqrt{4-y}$

$\Rightarrow$ $x^2 = 4 - y$

$y = 4 - x^2$, a parabola.

Here, the limits 1 and $\sqrt{4-y}$ are for x, 0 and 3 are for y.

When $x = 1$, on $y = 4 - x^2$, $y = 3$

Now, $\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy = \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx$

(By changing the order of integration)

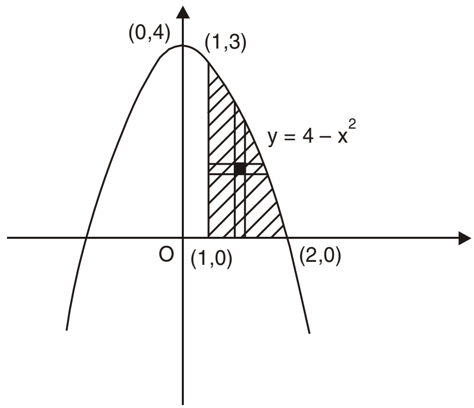

Fig. 3.14

$$= \int_1^2 \left[xy + y^2/2\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left(4x - x^3 + 8 - 4x^2 + \frac{x^4}{2}\right) dx$$

$$= \left[2x^2 - \frac{x^4}{4} + 8x - \frac{4}{3}x^3 + \frac{x^5}{10}\right]_1^2$$

$$= 6 - \frac{15}{4} + 8 - \frac{28}{3} + \frac{31}{10} = \frac{241}{60}.$$

Type 3. Evaluation by changing into polars

1. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ *by changing to polar coordinates.*

Solution. In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$

Since x, y varies from 0 to ∞

r also varies from 0 to ∞

In the first quadrant 'θ'

varies from 0 to $\pi/2$

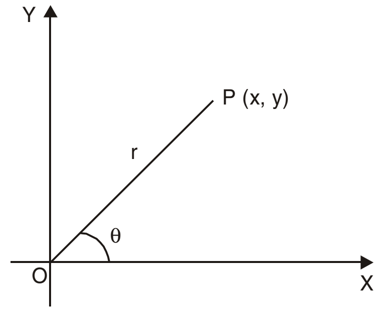

Fig. 3.15

Thus $$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

Put $$r^2 = t \quad \therefore\ r\, dr = \frac{dt}{2}$$

t also varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} (0-1)\, d\theta$$

$$= +\frac{1}{2} \int_0^{\pi/2} 1 \,.\, d\theta$$

$$= \frac{+1}{2} [\theta]_0^{\pi/2} = \frac{+1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

2. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ *by changing into polars.*

Solution $$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$$

$x = \sqrt{a^2 - y^2}$ or $x^2 + y^2 = a^2$ is a circle with centre origin and radius a. Since, y varies from 0 to a the region of integration is the first quadrant of the circle.

In polars, we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $$x^2 + y^2 = r^2$$

i.e., $$r^2 = a^2$$

$\Rightarrow$ $$r = a$$

Also $x = 0$, $y = 0$ will give $r = 0$ and hence we can say that r varies from 0 to a. In the first quadrant θ varies from 0 to $\pi/2$, we know that $dx\, dy = r\, dr\, d\theta$

$$\therefore \quad I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r \sin\theta \, r \, r \, dr \, d\theta$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta \, dr \, d\theta$$

$$= \int_{r=0}^{a} r^3 \left(-\cos\theta\right)_0^{\pi/2} dr$$

$$= \int_0^a -r^3 (0-1) \, dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

$$I = \frac{a^4}{4}.$$

Type 4. Applications of double and triple integrals

1. *Find the area of the circle $x^2 + y^2 = a^2$ by using double integral.*

Solution

Since, the circle is symmetric about the coordinates axes, area of the circle is 4 times the area *OAB* as shown in Figure.

For the region *OAB*, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a.

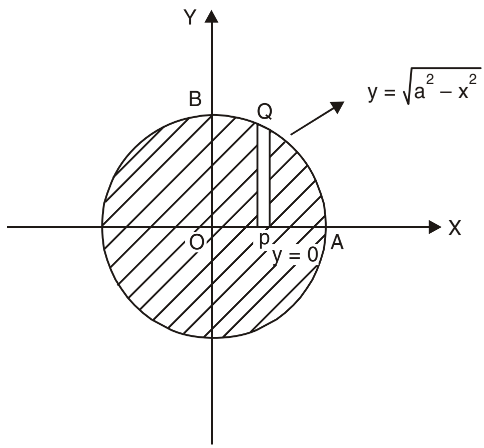

Fig. 3.16

$$\therefore \text{ Area of the circle } = 4 \int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} dy \, dx$$

$$= 4 \int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_0^a \sqrt{a^2 - x^2} \, dx$$

$$= 4 \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \pi a^2 \text{ sq. units}$$

2. *Find by double integration the area enclosed by the curve $r = a\,(1 + \cos\theta)$ between $\theta = 0$ and $\theta = \pi$.*

Solution Area $= \iint r \, dr \, d\theta$

where r varies from 0 to $a\,(1 + \cos\theta)$ and θ varies from 0 to π

$$\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

i.e., $$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_{0}^{\pi} \left\{2\cos^2\left(\frac{\theta}{2}\right)\right\}^2 d\theta$$

$$= 2a^2 \int_{0}^{\pi} \cos^4\left(\frac{\theta}{2}\right) d\theta$$

Put $\theta/2 = \phi,\ d\theta = 2d\phi$

and ϕ varies from 0 to $\pi/2$

$\therefore$ $$A = 2a^2 \int_{0}^{\pi/2} \cos^4 \phi \cdot 2d\phi$$

$$= 4a^2 \int_{0}^{\pi/2} \cos^4 \phi \cdot d\phi$$

$$= 4a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{(by the reduction formula)}$$

Area, $A = 3\pi a^2/4$ sq. units.

3. *Find the value of* $\iiint z\, dx\, dy\, dz$ *where V is the hemisphere* $x^2 + y^2 + z^2 = a^2,\ z \geq 0.$

Solution

Let $$I = \iiint_V z\, dx\, dy\, dz$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$1+\cos\theta = 2\cos^2\frac{\theta}{2}$$

$$= \frac{1}{2} \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(a^2 - x^2 - y^2\right) dy\, dx$$

$$= \frac{1}{2} \int_{x=-a}^{a} \left[\left(a^2 - x^2\right) y - \frac{y^3}{3}\right]_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_{-a}^{a} \left(a^2 - x^2\right)^{3/2} dx$$

$$= \frac{2}{3} . 2 \int_{0}^{a} \left(a^2 - x^2\right)^{3/2} dx$$

Put $x = a \sin \theta$

$dx = a \cos \theta \, d\theta$

θ varies from 0 to $\pi/2$

$$= \frac{4}{3} \int_{\theta=0}^{\pi/2} \left(a^2 \cos^2 \theta\right)^{3/2} a \cos\theta \, d\theta$$

$$= \frac{4a^4}{3} \int_{0}^{\pi/2} \cos^4 \theta \, d\theta$$

$$= \frac{4a^4}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \qquad \text{(By applying reduction formula)}$$

$$= \frac{\pi a^4}{4}$$

Thus, $I = \frac{\pi a^4}{4}$.

4. *Using multiple integrals find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

The volume (V) is 8 times in the first octant (V_1)

i.e., $V = 8V_1 = 8 \iiint dz\, dy\, dx$

z varies from 0 to $c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to $(b/a)\ \sqrt{a^2 - x^2}$

x varies from 0 to a

$$V = 8V_1 = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2 - x^2}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2 - x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2 - x^2}} \frac{1}{b}\sqrt{b^2\left\{1-\left(\frac{x^2}{a^2}\right)\right\} - y^2}\, dy\, dx$$

We shall use $\int \sqrt{\alpha^2 - y^2}\, dy = \dfrac{y\sqrt{\alpha^2 - y^2}}{2} + \dfrac{\alpha^2}{2}\sin^{-1}\left(\dfrac{y}{\alpha}\right)$

where $\alpha^2 = b^2\ \{1 - x^2/a^2\} = b^2\ (a^2 - x^2)/a^2$

$$\therefore \qquad V = \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{\alpha} \sqrt{\alpha^2 - y^2}\, dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\frac{y\sqrt{\alpha^2 - y^2}}{2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{y}{\alpha}\right)\right]_0^{\alpha} dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} 0 + \frac{\alpha^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \frac{\pi}{2}\cdot\frac{1}{2}\frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$= \frac{2bc\pi}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^{a}$$

$$= \frac{2bc\pi}{a^2}\cdot\frac{2a^3}{3} = \frac{4\pi\, abc}{3}$$

Thus the required volume $(V) = \dfrac{4\pi abc}{3}$ cubic units.

EXERCISE 3.2

1. Evaluate $\iint_R xy^2\,dx\,dy$ over the region bounded by $y = x^2$, $y = 0$ and $x = 1$. $\left[\textbf{Ans.}\ \dfrac{1}{24}\right]$

2. Evaluate $\iint_R xy\,(x + y)\,dx\,dy$ taken over the region bounded by the parabolas $y^2 = x$ and $y = x^2$. $\left[\textbf{Ans.}\ \dfrac{3}{28}\right]$

3. Evaluate $\iint_R x^2 y\,dx\,dy$ over the region bounded by the curves $y = x^2$ and $y = x$. $\left[\textbf{Ans.}\ \dfrac{1}{35}\right]$

4. Evaluate $\iint_R xy\,dx\,dy$ where R is the region in the first quadrant bounded by the line $x + y = 1$. $\left[\textbf{Ans.}\ \dfrac{1}{6}\right]$

Evaluate the following by changing the order of integration (5 to 9)

5. $\displaystyle\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} \left(x^2 + y^2\right) dy\,dx.$ $\left[\textbf{Ans.}\ \dfrac{a^3}{28} + \dfrac{a}{20}\right]$

6. $\displaystyle\int_0^a \int_0^{2\sqrt{ax}} x^2\,dx\,dy.$ $\left[\textbf{Ans.}\ \dfrac{4a^4}{7}\right]$

7. $\displaystyle\int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} (a - x)\,dy\,dx.$ $\left[\textbf{Ans.}\ \dfrac{\pi a^3}{2}\right]$

8. $\displaystyle\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}.$ $\left[\textbf{Ans.}\ \dfrac{\pi a^2}{6}\right]$

9. $\displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a - x} xy\,dy\,dx.$ $\left[\textbf{Ans.}\ \dfrac{3a^4}{8}\right]$

10. Evaluate $\displaystyle\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\,dy\,dx$ by transforming into polar coordinates. $\left[\textbf{Ans.}\ \dfrac{5\pi a^4}{8}\right]$

11. Find the area of the cardioid $r = a\,(1 + \cos\theta)$ by double integration. $\left[\textbf{Ans.}\ \dfrac{3\pi\, a^2}{2}\right]$

12. Find the volume of the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = 3$. **[Ans.** 48π**]**