POISSON DISTRIBUTION

It is a discrete distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. (i.e. $n \to \infty$ and $p \to 0$)

e.g: The number of person born blind per year in a large city.
The number of death by horse kick in an army corps.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed ($= \lambda$ say)

Derive Poisson distribution as a limiting case of Binomial distribution

The Probability of r successes in a binomial distribution is

$$P(r) = {}^nC_r\, p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-(r-1))\,(n-r)!}{r!\,(n-r)!}\, p^r q^{n-r}$$

$$= \frac{(1-1/n)(1-2/n)\cdots(1-\frac{r-1}{n})}{r!}\, n^r p^r (1-p)^{n-r}$$

$$= \frac{(1-1/n)(1-2/n)\cdots(1-\frac{r-1}{n})(np)^r(1-\frac{np}{n})^{n-r}}{r!}$$

taking $np = \lambda$, $$P(r) = \frac{(1-1/n)(1-2/n)\cdots(1-\frac{r-1}{n})\,\lambda^r\,(1-\frac{\lambda}{n})^{n-r}}{r!}$$

taking $n \to \infty$ and $p \to 0$ $$P(r) = \frac{\lambda^r}{r!}\,\frac{(1-\lambda/n)^n}{(1-\lambda/n)^r} \text{ as } n \to \infty$$

$$P(r) = \frac{\lambda^r}{r!}\left(1 - n\lambda/n + \frac{n(n-1)}{2!}\frac{\lambda^2}{n^2} - \cdots\right) \times \frac{1}{(1-\lambda/n)^r} \text{ as } n \to \infty$$

$$P(r) = \frac{\lambda^r}{r!}\left(1 - \lambda + \frac{\lambda^2}{2!} - \cdots\right) \quad \therefore \boxed{P(r) = \frac{\lambda^r}{r!}e^{-\lambda}} \text{ Imp}$$

This is Poisson distribution.

The probabilities of 0, 1, 2, ..., r successes in a Poisson distribution are given by $e^{-\lambda}, \lambda e^{-\lambda}, \frac{\lambda^2}{r!}e^{-\lambda}, \ldots, \frac{\lambda^r}{r!}e^{-\lambda}$ respectively. The sum of the probabilities is unity.

Constants of Poisson Distribution

Constants of Poisson distribution can be obtained from the constants of Binomial distribution by taking $n \to \infty, p \to 0$ and $q \to 1$

Mean $= \lim_{n\to\infty} np = \lim_{n\to\infty} \lambda = \lambda$

S.d $= \lim_{n\to\infty} \sqrt{npq} = \lim_{n\to\infty} \sqrt{\lambda} = \sqrt{\lambda}$

Variance $\mu_2 = \lambda$, Skewness $= 1/\sqrt{\lambda}$, Kurtosis $= 3 + \frac{1}{\lambda}$

eg1. A manufacturer of cotterpins knows that 5% of his product is defective. If he sells pins in boxes of 100 and gaurantees that no more than 10 pins will be defective. What is the approximate probability that a box will fail to meet the gauranteed quality [$e^{-5} = 0.006738$]

Soln

p = Probability of defective $= 0.05$

$n = 100$

$np = 5 = \lambda$

$$P(r) = \frac{\lambda^r}{r!}e^{-\lambda}$$

Probability of more than ten will be defective.

$$= 1 - [P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)+P(8)+P(9)+P(10)]$$

$$= 1 - [e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^3}{3!}e^{-\lambda} + \frac{\lambda^4}{4!}e^{-\lambda} + \frac{\lambda^5}{5!}e^{-\lambda} + \frac{\lambda^6}{6!}e^{-\lambda} + \frac{\lambda^7}{7!}e^{-\lambda} + \frac{\lambda^8}{8!}e^{-\lambda} + \frac{\lambda^9}{9!}e^{-\lambda} + \frac{\lambda^{10}}{10!}e^{-\lambda}]$$

$$= 1 - \sum_{r=0}^{10} \frac{e^{-5}\,5^r}{r!} = 0.013695 \text{ Ans.}$$

e.g. 2 A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of day (i) on which there is no demand (ii) on which demand is refused.

Soln Given mean $= \lambda = 1.5$

Poisson distribution $P(r) = \frac{\lambda^r}{r!} e^{-\lambda}$

(i) Proportion of day on which there is no demand $= e^{-1.5} = 0.2231$

(ii) Proportion of day on which demand is refused

i.e. Probability of demand to be more than 2

$= 1 - [P(0) + P(1) + P(2)] = 1 - [e^{-1.5} + 1.5e^{-1.5} + \frac{(1.5)^2}{2!} e^{-1.5}]$

$= 1 - e^{-1.5}(1 + 1.5 + 1.125) = 1 - 0.2231(1 + 1.5 + 1.125)$

$= 0.1912625$ Ans

e.g. 3 In a certain factory turning out razor blades, there is small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Using Poisson's distribution to calculate the approximate number of packets containing no defect, one defect and two defective blades respectively in a consignment of 10,000 packets.

Soln Given $n = 10$

Probability of defective $= p = 0.002$

$\lambda = np = 10 \times 0.002 = 0.02$

Probability of no defect $= P(0) = e^{-\lambda} = e^{-.02} = 0.9802$

Probability of two defect $= P(2) = \frac{\lambda^2}{2!} e^{-\lambda} = \frac{(.02)^2}{2!} e^{-.02} = \frac{(.02)^2}{2!}(.9802)$

Probability of one defect $= P(1) = \lambda e^{-\lambda} = (.02)(.9802)$

∴ Reqd number of packets of no defect in 10,000 packets $= 9802$ pkts.

Reqd number of packets of one defect $(.02)(.9802) \times 10{,}000 = 196$ pkts

Reqd number of packets of two defect $\frac{(.02)^2}{2!}(.9802) \times 10{,}000 = 2$ pkts Approx eg 4. Assume that the probability of an individual coalminer being killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be atleast one fatal accident in a year.

Soln Given $n=200$, $p=\frac{1}{2400}$ $\therefore \lambda = np = 0.083$

Probability of atleast one fatal accident $= 1-P(0)$

$= 1-e^{-\lambda} = 1-e^{-.083}$

$= 0.08$ $(\because e^{-.083} = .92)$

Recurrence formula for Poisson distribution

$P(r) = \frac{\lambda^r}{r!}e^{-\lambda}$, $P(r+1) = \frac{\lambda^{r+1}}{(r+1)!}e^{-\lambda}$

$\frac{P(r+1)}{P(r)} = \left[\frac{\lambda^{r+1}}{(r+1)!}e^{-\lambda}\right] \Big/ \frac{\lambda^r}{r!}e^{-\lambda} = \frac{\lambda}{r+1}$

$\therefore \boxed{P(r+1) = \frac{\lambda}{r+1}P(r)}$

eg 5. If the variance of Poisson distribution is 2, find the probabilities for $r = 1, 2, 3, 4$ from recurrence relation of Poisson distribution.

Soln Given variance $= 2 = \lambda$

$P(0) = \frac{\lambda^0}{0!}e^{-2} = 0.1353$

$P(1) = \frac{\lambda}{1}P(0) = 2\times 0.1353 = 0.2706$

$P(2) = \frac{\lambda}{2}P(1) = 0.2706$

$P(3) = \frac{\lambda}{3}P(2) = \frac{2}{3}\times 0.2706 = 0.1804$

$P(4) = \frac{\lambda}{4}P(3) = \frac{2}{4}\times(0.1804) = 0.0902$

Fitting of Poisson distribution: To find expected frequency of the given data

$$\boxed{N \frac{\lambda^r}{r!} e^{-\lambda}}$$ N = Total frequency

Eg 6 Data was collected over a period of 10 years, showing number of death from horse kicks in each of the 20 army Corps. The distribution of death was as follows

No of death:	0	1	2	3	4
frequency:	109	65	22	3	1

Fit a poisson distribution to the data and calculate the theoretical frequencies

Sol

We have formula for fitting $N \frac{\lambda^r}{r!} e^{-\lambda}$, where λ = mean, N = total frequency = 200

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{(0\times 109)+(1\times 65)+(2\times 22)+(3\times 3)+(4\times 1)}{200} = 0.61$$

When $r=0$, $N\frac{\lambda^r}{r!}e^{-\lambda} = 200 \times e^{-.61} = 200 \times 0.5435 = 108.7$

$r=1$, $= \frac{200\times 0.61}{1!} e^{-.61} = 108.7 \times 0.61 = 66.3$

Illy $r=2$, $f_e = 20.2$, $r=3$, $f_e = 4.1$, $r=4$, $f_e = 0.7$

∴

No of death	0	1	2	3	4
f_o :	109	65	22	3	1
f_e :	109	66	20	4	1

Ans.

Q.7. Fit a Poisson's distribution to the following data and Calculate theoretical frequencies

x:	0	1	2	3	4
f:	122	60	15	2	1

Sol:

Given $N = 200$, $\lambda = \frac{\Sigma f_i x_i}{\Sigma f_i} = 0.5$

∴ Using Poisson distribution formula i.e. $N \frac{\lambda^r}{r!} e^{-\lambda}$

$r=0$, $N \frac{\lambda^r}{r!} e^{-\lambda} = 200 \times \frac{(0.5)^0}{0!} e^{-0.5} = 200 \times 0.61 = 122$

$r=1$ $N \frac{\lambda^r}{r!} e^{-\lambda} = 200 \times \frac{(0.5)}{1!} e^{-0.5} = 122 \times 0.5 = 61$

Illy $r=2$ $fe = 15.2$, $r=3$, $fe = 1.25$, $r=4$, $fe = 2.54$

∴

x:	0	1	2	3	4
f_o:	122	60	15	2	1
fe:	122	61	15	1	3