PROBABILITY DISTRIBUTION

- Probability Distribution
  - Discrete Probability Distribution
    - Binomial Distribution
    - Poisson Distribution
  - Continuous Probability Distribution
    - Normal Distribution.

BINOMIAL DISTRIBUTION (Bernoulli's distribution)

Let n = Total number of repeated trials
p = probability of success
q = Probability of failure

Then the probability of r successes in a series of n trials is given by

$^nC_r p^r q^{n-r}$, where r takes the integral value from 0 to n.

The probabilities of $0, \quad 1, \quad 2, \ldots, r, \ldots, n$ successes are therefore given by $^nC_0 p^0 q^n, \; ^nC_1 p^1 q^{n-1}, \; ^nC_2 p^2 q^{n-2}, \ldots \; ^nC_r p^r q^{n-r}, \ldots \; ^nC_n p^n q^{n-n}$ respectively.

or $q^n, \; ^nC_1 p q^{n-1}, \; ^nC_2 p^2 q^{n-2}, \ldots, \; ^nC_r p^r q^{n-r}, \ldots, p^n$

The probability of the number of successes so obtained is called Binomial Distribution.

The above probabilities are the successive terms of the binomial expansion of $(q+p)^n$; that is why it is called Binomial Distribution.

The sum of the probabilities is

$$q^n + {}^nC_1 p q^{n-1} + {}^nC_2 p^2 q^{n-2} + \cdots + {}^nC_r p^r q^{n-r} + \cdots + p^n = (q+p)^n = 1$$

The probabilities of atleast r successes in n trials is

$$^nC_r p^r q^{n-r} + {}^nC_{r+1} p^{r+1} q^{n-(r+1)} + \cdots + p^n$$

or $$1 - \{ {}^nC_0 p^0 q^n + {}^nC_1 p^1 q^{n-1} + \cdots + {}^nC_{r-1} p^{r-1} q^{n-(r-1)} \}.$$

Constants of Binomial distribution

Mean $= \mu_1' = np$, variance: $\mu_2 = npq$ s.d $= \sqrt{npq}$

$\mu_3 = npq(q-p)$, $\mu_4 = npq[1+3(n-2)pq]$

Skewness $(\sqrt{\beta_1})$, $\beta_1 = \frac{1-2p}{\sqrt{npq}}$, Kurtosis: $\beta_2 = 3 + \frac{1-6pq}{npq}$

e.g. 1. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples how many would be expected to contain at least 3 defective parts.

Soln Let p = probability of defective parts

n = total number of parts in a sample = 20

mean of the defective parts = 2

i.e. $np = 2 \Rightarrow 20 \cdot p = 2 \Rightarrow p = \frac{2}{20} = 0.1$

$p = 0.1$, $q = 0.9$

Probability of atleast three defective parts in the sample will be

$$1 - [P(0)+P(1)+P(2)] = 1 - [{}^{20}C_0(.1)^0(.9)^{20} + {}^{20}C_1(.1)(.9)^{19} + {}^{20}C_2(.1)^2(.9)^{18}]$$

$$= 1 - (.9)^{18} \times 4.51 = 0.323$$

out of 1000 such samples the expected number of parts containing atleast 3 defective parts $= 0.323 \times 1000 = 323$ Ans

H.A (1) Assume that on the average one telephone number out of fifteen called between 2 P.M and 3 P.M. on week-days is busy. What is the probability that if 6 randomly selected telephone number are called (i) not more than three (ii) atleast three of them will be busy; [Ans (i) 0.9997, (ii) 0.005]

(2) If on an average one ship in every ten is wrecked, find the probability that out of 5 ship expected to arrive, 4 atleast will arrive safely. [Ans: 0.91854]

FITTING OF BINOMIAL DISTRIBUTION

To find the theoretical frequencies against the given observed frequencies is generally termed as fitting.

To fit in Binomial distribution the formula for finding theoretical frequencies or expected frequency is given by

$\boxed{N\, {}^nC_x\, p^x q^{n-x}}$ N = Sum of the frequencies.

eg 2. The following data are the number of seeds germinating out of 10 on a damp filter paper for 80 sets of seeds. Fit a Binomial distribution to these data

[Fit a Binomial distribution to the following data]

x:	0	1	2	3	4	5	6	7	8	9	10
f:	6	20	28	12	8	6	0	0	0	0	0

Solⁿ Given $n = 10,\ N = 80,\ p = ?,\ q = ?$

formula $N\, {}^nC_x\, p^x q^{n-x}$

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{20 + (2\times28) + (3\times12) + (4\times8) + (5\times6)}{80} = \frac{174}{80}$$

Mean for binomial distribution $= np \Rightarrow 10\times p = \frac{174}{80} \Rightarrow p = \frac{174}{80\times10} = 0.2175$

$p = 0.2175,\quad q = 1 - p = 0.7825$

When $x=0$, $N\, {}^nC_x\, p^x q^{n-x} = 80\times(.2175)^0(.7825)^{10} = 6.89$

When $x=1$, $N\, {}^nC_x\, p^x q^{n-x} = 80\times {}^{10}C_1 (.2175)^1(.7825)^9 = 19.14$

When $x=2$, $N\, {}^nC_x\, p^x q^{n-x} = 80\times {}^{10}C_2 (.2175)^2(.7825)^8 = 23.94$

Hly we can find the other values

x:	0	1	2	3	4	5	6	7	8	9	10
fe:	6.89	19.14	23.94	17.74	8.63	2.88	0.67	0.1	.01	.00	00

Ans.

eg. 3 Fit a binomial distribution to the following data

x:	0	1	2	3	4	5
f:	2	14	20	34	22	8

Soln formula $N\,{}^{n}C_{x}\,p^{x}q^{n-x}$

Given $N = 100$, $n = 5$

$$\text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{(0\times2)+(1\times14)+(2\times20)+(3\times34)+(4\times22)+(5\times8)}{100} = 2.84$$

mean in binomial dist. $= np \Rightarrow 5\times p = 2.84 \Rightarrow p = .568,\ q = 0.432$

When $x=0$, $N\,{}^{n}C_{x}\,p^{x}q^{n-x} = 100\times{}^{5}C_{0}\,(.568)^{0}(0.432)^{5} = 1.5$

When $x=1$, $N\,{}^{n}C_{x}\,p^{x}q^{n-x} = 100\times{}^{5}C_{1}\,(.568)^{1}(0.432)^{4} = 9.89$

etc.

When $x=2$, $f_e = 26$; $x=3$, $f_e = 34.2$; $x=4$, $f_e = 22.4$; $x=5$, $f_e = 5.9$

$\therefore$

x:	0	1	2	3	4	5
f_o:	2	14	20	34	22	8
f_e:	1.5	9.89	26	34.2	22.4	5.9

Ans.

Recurrence formula for Binomial distribution

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{(n-r)!\,r!} p^r q^{n-r}$$

$$P(r+1) = {}^nC_{r+1} p^{r+1} q^{n-(r+1)} = \frac{n!}{(n-(r+1))!\,(r+1)!} p^{r+1} q^{n-(r+1)}$$

$$\frac{P(r+1)}{P(r)} = \frac{n!/(n-(r+1))!\,(r+1)!}{n!/(n-r)!\,r!} \quad \frac{p^{r+1} q^{n-(r+1)}}{p^r q^{n-r}}$$

$$= \frac{(n-r)(n-r-1)!\,r!}{(n-r-1)!\,(r+1)r!} \cdot \frac{p}{q}$$

$$\therefore \boxed{P(r+1) = \frac{n-r}{r+1} \frac{p}{q} P(r)}$$