Singular point, Poles and Residues

1. A point at which a function $f(z)$ ceases to be analytic is called Singular point or Singularity of $f(z)$.
   e.g $z=2$ is a singular point of $f(z)=\frac{1}{z-2}$

2. Isolated Singular point
   A Singular point $z=a$ of a function $f(z)$ is called an isolated singular point if there exists a circle with centre a which contains no other singular point of $f(z)$.
   e.g. $z=-1, 1$ are two isolated singular points of the function $f(z)=\frac{z}{z^2-1}$
   The function $f(z)=\frac{1}{\sin(\pi z)}$ has infinite number of isolated singular points at $z=\pm 1, \pm 2 \ldots$

3. When $z=a$ is an isolated singular point of $f(z)$, we can expand $f(z)$ in a Laurent series about $z=a$
   $f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\cdots+a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+\cdots$
   If $a_{-n}\neq 0$ and $a_{-n-1}=a_{-n-2}=\cdots=0$, then $z=a$ is called a pole of order n.
   $\left[\text{Where } a_n=\frac{1}{2\pi i}\int_c \frac{f(w)}{(w-a)^{n+1}}\,dw\right]$

4. If all the negative powers of $(z-a)$ after nth are missing, then $z=a$ is called a pole of order n. A pole of order one is called simple pole
   If the number of negative power of $(z-a)$ is infinite then $z=a$ is called an essential singularity.

e.g $\frac{1}{(z-2)^3}$ i.e $(z-2)^{-3}$ i.e. after the negative power of 3 are missing, then $z=2$ is called a pole of order 3.

5. Residues: The Coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the Residue of $f(z)$ at that point.

Thus the residue of $f(z)$ at $z=a$ is a_{-1}

i.e. $a_{-1} = \frac{1}{2\pi i}\int_c f(z)\,dz$ or $\int_c f(z)\,dz = 2\pi i\, a_{-1}$

i.e.

$$\boxed{\int_c f(z)\,dz = 2\pi i \,\mathrm{Res}\, f(a)}$$ Imp.

Residue Theorem:

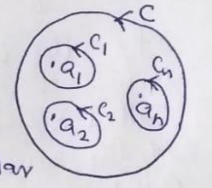
If $f(z)$ is analytic at all points inside and on a simple closed curve C, except at a finite number of singular points within c then

$$\int_C f(z)\,dz = 2\pi i\left[\text{sum of the residues at singular points within } C\right]$$

Symbolically

$$\boxed{\int_C f(z)\,dz = 2\pi i \,\Sigma\, \mathrm{Res}}$$ Imp.

Proof: Let C be a simple closed curve where $f(z)$ is analytic except the finite number of singular points $a_1, a_2, \dots, a_n$.



Let us draw small circles $C_1, C_2, \dots, C_n$ enclosing $a_1, a_2, \dots a_n$ respectively such that there should not be any other singular point. Then from Cauchy Integral theorem

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$$

$$= 2\pi i\left[\mathrm{Res}\, f(a_1) + \mathrm{Res}\, f(a_2) + \dots + \mathrm{Res}\, f(a_n)\right]$$

$$\int_C f(z)\,dz = 2\pi i \;\Sigma\, \mathrm{Res}$$

hence proved

Calculation of Residues

- If $f(z)$ has a simple pole (i.e. pole of order one) at $z=a$, then

$$\boxed{\operatorname{Res} f(a) = \lim_{z\to a}\left[(z-a)f(z)\right]}$$

- If $f(z)$ has a pole of order n at $z=a$ then

$$\boxed{\operatorname{Res} f(a) = \frac{1}{(n-1)!}\lim_{z\to a}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}}$$

e.g. 1. Determine the poles of the function $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each point.

Sol. $f(z)$ has one simple pole (i.e. order one) at $z=-2$ and a pole of order 2 at $z=1$

$$\operatorname{Res} f(-2) = \lim_{z\to -2}\left[(z+2)f(z)\right] = \lim_{z\to -2}\left[(z+2)\frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to -2}\left[\frac{z^2}{(z-1)^2}\right] = \frac{4}{9}$$

$\therefore \operatorname{Res} f(-2) = \dfrac{4}{9}$ Ans

$$\operatorname{Res} f(1) = \frac{1}{(2-1)!}\lim_{z\to 1}\left[\frac{d}{dz}(z-1)^2 f(z)\right] = \lim_{z\to 1}\left[\frac{d}{dz}(z-1)^2\frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\frac{z^2}{(z+2)}\right] = \lim_{z\to 1}\frac{(z+2)\times 2z - z^2}{(z+2)^2} = \lim_{z\to 1}\frac{z^2+4z}{(z+2)^2}$$

$\therefore \operatorname{Res} f(1) = \dfrac{5}{9}$ Ans e.g.2 Evaluate $\int_C \frac{z-2}{z^2-z}\,dz$, where $C: |z| = 2$.

Soln Let $f(z) = \frac{z-2}{z^2-z} = \frac{z-2}{z(z-1)} = \frac{z-2}{(z-0)(z-1)}$

$f(z)$ has two simple poles at $z = 0, 1$ within $C: |z| = 2$

$\text{Res} f(0) = \lim_{z\to 0} [(z-0) f(z)] = \lim_{z\to 0} \left[z \cdot \frac{z-2}{z(z-1)}\right] = \lim_{z\to 0} \frac{z-2}{z-1} = 2$

$\text{Res } f(0) = 2$

$\text{Res} f(1) = \lim_{z\to 1} [(z-1) f(z)] = \lim_{z\to 1} \left[(z-1) \frac{z-2}{z(z-1)}\right] = \lim_{z\to 1} \frac{z-2}{z} = -1$

$\text{Res} f(1) = -1$

hence by Residue theorem $\int_C \frac{z-2}{z^2-z}\,dz = 2\pi i \sum \text{Res} = 2\pi i (2-1)$

$\int_C \frac{z-2}{z^2-z}\,dz = 2\pi i$ Ans

e.g 3. Evaluate $\int_C \frac{2z-1}{z(z+1)(z-3)}\,dz$, where C is the circle $|z| = 2$

Soln $f(z) = \frac{2z-1}{z(z+1)(z-3)}$, has three simple poles at $z = 0, -1, \& 3$. out of which $0, -1$, lies inside the circle $|z| = 2$ and $z = 3$ lies outside the circle $|z| = 2$

$\therefore$ we require to find residues at $z = 0, \& -1$

$\text{Res} f(0) = \lim_{z\to 0} [(z-0) f(z)] = \lim_{z\to 0} \left[(z-0) \frac{2z-1}{z(z+1)(z-3)}\right] = \lim_{z\to 0} \frac{2z-1}{(z+1)(z-3)}$

$\therefore \text{Res} f(0) = \frac{1}{3}$

$\text{Res} f(-1) = \lim_{z\to -1} [(z+1) f(z)] = \lim_{z\to -1} \left[(z+1) \frac{2z-1}{z(z+1)(z-3)}\right] = \lim_{z\to -1} \frac{2z-1}{z(z-3)}$

$\therefore \text{Res} f(-1) = -\frac{3}{4}$.

hence by Residue theorem $\int_C \frac{2z-1}{z(z+1)(z-3)}\,dz = 2\pi i \sum \text{Res} = 2\pi i \left[\frac{1}{3} - \frac{3}{4}\right]$

$= 2\pi i \,(-5/12) = -5\pi i/6$ Ans

e.g.4 Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is the circle

(i) $C: |z+1+i| = 2$

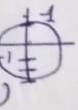
(ii) $C: |z+1-i| = 2$

Soln Let $f(z) = \frac{z-3}{z^2+2z+5}$

$z^2+2z+5=0 \Rightarrow z = \frac{-2 \pm \sqrt{2^2-4.1.5}}{2\cdot 1} \Rightarrow z = -1 \pm 2i$

(i) $f(z)$ has two simple poles at $z=-1+2i$, $z=-1-2i$ out of which $z=-1-2i$ lies in the circle $C: |z-(-1-i)|=2$ and $z=-1+2i$ lies outside the circle

$\therefore$ $\text{Res}\, f(-1-2i) = \lim_{z \to -1-2i} \left[\frac{z-(-1-2i)\; z-3}{(z-(-1-2i))(z-(-1+2i))} \right]$



$= \lim_{z \to -1-2i} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \frac{i+2}{2i}$

$\therefore \int_C f(z)\,dz = 2\pi i \sum \text{Res} = 2\pi i \frac{(i+2)}{2i} = (2+i)\pi$ Ans.

(ii) $f(z)$ has two simple poles at $z=-1+2i$, $z=-1-2i$ out of which $z=-1+2i$ lies inside the circle and $z=-1-2i$ lies outside the circle $C: |z-(-1+i)|=2$ because centre is $-1+i$

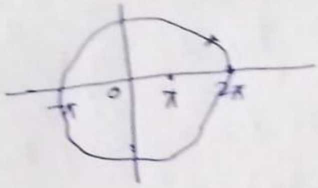
$\text{Res}\, f(-1+2i) = \lim_{z \to -1+2i} \frac{(z-(-1+2i))(z-3)}{z^2+2z+5} = \lim_{z \to -1+2i} \frac{(z-3)(z-(-1+2i))}{(z-(-1-2i))\,(z-(-1+2i))}$

$= \lim_{z \to -1+2i} \frac{z-3}{z+1+2i} = \frac{2i-4}{4i} = \frac{i-2}{2i}$

$\therefore \int_C f(z)\,dz = 2\pi i \sum \text{Res} = 2\pi i \frac{i-2}{2i} = (i-2)\pi$ Ans.

Let $f(z) = h(z)/g(z)$, where h is continuous at z_0 and $h(z_0) \neq 0$. Let g is differentiable at z_0 and $f(z)$ has a simple pole at z_0 then $\text{Res } f(z_0) = \dfrac{h(z_0)}{g'(z_0)}$

Q15. Evaluate $\oint_C \dfrac{4iz-1}{\sin(z)} dz$, where C is a closed path enclosing $-\pi$ to 2π



Sol. Let $f(z) = \dfrac{4iz-1}{\sin z}$

Singular points $-\pi, 0, \pi, 2\pi$

$$\text{Res } f(0) = \left(\frac{4iz-1}{\cos z}\right)_{z=0} = -1$$

$$\text{Res } f(-\pi) = \frac{-4i\pi-1}{\cos(-\pi)} = 1+4\pi i$$

$$\text{Res } f(\pi) = \frac{4\pi i-1}{\cos\pi} = 1-4\pi i$$

$$\text{Res } f(2\pi) = \frac{4i\times 2\pi-1}{\cos(2\pi)} = 8\pi i-1$$

$$\therefore \int_c f(z)\,dz = 2\pi i \sum \text{Res} = 2\pi i \times 8\pi i = -16\pi^2 \text{ Ans}$$