Power Series: A series of the form

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots = \sum_{m=0}^{\infty} c_m (z-a)^m \quad \cdots (1)$$

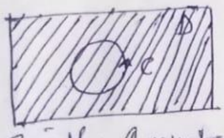
is called a power series in power of $(z-a)$. where z is a variable, $c_0, c_1, c_2, \ldots$ are constants called the coefficients and a is a constant called the centre of the series.

If the series (1) is convergent for all values of z satisfying $|z-a|<R$, ie for all values of z lying inside the circle with centre at $z=a$ and radius R then the circle $|z-a|=R$ is called the circle of convergence and R is called the radius of convergence.

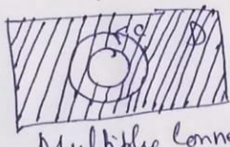
Note: 1 Simply connected and Multiply connected region

A domain D is said to be simply connected if every closed curve in D includes the points of D (except self intersection) otherwise multiply connected.

e.g.: • Simply connected: circular disc, ellipse, etc.
• Multiply connected: ring shaped domain.



Simply connected

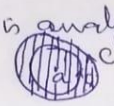


Multiply connected

Self intersection

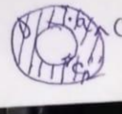
Note. 2 • In simply connected region D, $f(z)$ is analytic, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$



• In Multiply connected region D, $f(z)$ is analytic in D then

$$f(a) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-a} dz,$$ where both integral are taken counter clockwise sense



Taylor's Series: If $f(z)$ is analytic inside a circle C with centre a, then for z inside C

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

Laurent's Series: If $f(z)$ is analytic in the ring shaped region D, bounded by two concentric circles C_1 & C_2 of radii r_1, r_2 $(r_1 > r_2)$ and with centre at a, then for all z in D

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

i.e

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,dw$, C being any curve in D enclosing C_2

Eg 1 Expand $\frac{1}{z^2-3z+2}$ in the region (a) $|z|<1$ (b) $1<|z|<2$

Sol Here $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

(a) When $|z|<1$, then $\left|\frac{z}{2}\right|<1$

or $f(z) = \frac{1}{-2(1-z/2)} + \frac{1}{(1-z)} = \frac{1}{-2}(1-z/2)^{-1} + (1-z)^{-1}$

$\therefore f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) + (1 + z + z^2 + \cdots)$

$f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \cdots$ which is a Taylor Series

(b) When $1 < |z| < 2$, then $|z/2| < 1$ and $\left|\frac{1}{z}\right| < 1$

$\therefore f(z) = \dfrac{1}{-2(1 - z/2)} - \dfrac{1}{z(1 - 1/z)}$

$= -\frac{1}{2}(1 - z/2)^{-1} - \frac{1}{z}(1 - \frac{1}{z})^{-1}$

$= -\frac{1}{2}(1 + z/2 + \frac{z^2}{4} + \dots) - \frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \dots)$

$= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} + \dots - z^{-1} - z^{-2} - z^{-3} - \dots$

which is a Laurent Series

Eg 2 Obtain the Taylor's expansion of $\log(1+z)$, when $|z| < 1$

Soln Here $z = -1$ is the singular point

Region of validity of Taylor's Series $|z - 0| < 1$

Taylor's Series will be

$$f(z) = f(0) + (z-0) f'(0) + \frac{(z-0)^2}{2!} f''(0) + \dots \quad (1)$$

We have $f(z) = \log(1+z)$

$f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z} \Rightarrow f'(0) = 1$

$f''(z) = -\frac{1}{(1+z)^2} \Rightarrow f''(0) = -1$ and so on

$\therefore$ from (1)

$$\log(1+z) = 0 + z - \frac{z^2}{2!} + \dots$$ Ans.

e.g. 3. Find the Taylor and Laurent's series which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ when (i) $|z|<2$, (ii) $2<|z|<3$ (iii) $|z|>3$

Sol^n Let $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

(i) $|z|<2$, $\because |z/2|<1$ and $|z/3|<1$

$$\therefore f(z) = 1 + \frac{3}{2(1+z/2)} - \frac{8}{3(1+z/3)} = 1 + \frac{3}{2}(1+z/2)^{-1} - \frac{8}{3}(1+z/3)^{-1}$$

$$= 1 + \frac{3}{2}\left(1 - z/2 + (z/2)^2 - \cdots\right) - \frac{8}{3}\left(1 - z/3 + (z/3)^2 + \cdots\right)$$

$$= \left(1 + \frac{3}{2} - \frac{8}{3}\right) - \frac{3}{4}z + \frac{8}{9}z + \frac{3}{8}z^2 - \frac{8}{27}z^2 + \cdots$$

$$= -\frac{1}{6} + \frac{5}{36}z + \left(\frac{3}{8} - \frac{8}{27}\right)z^2 + \cdots$$ which is a Taylor's Series

(ii) $\left|\frac{2}{z}\right|<1$ and $|z/3|<1$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}(1+z/3)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left(1 - \frac{2}{z} + 4/z^2 + \cdots\right) - \frac{8}{3}\left(1 - z/3 + z^2/9 + \cdots\right)$$

$$f(z) = -5/3 + \frac{8}{9}z - \frac{8}{27}z^2 + \cdots + 3z^{-1} - 6z^{-2} + 12z^{-3} + \cdots$$

Which is a Laurent's Series

(iii) Let us assume that $3<|z|<R$, where R is very large

$\because |z|>3 \Rightarrow \left|\frac{3}{z}\right|<1$ and $|2/z|<1$

$$\therefore f(z) = 1 + \frac{3}{z}(1+2/z)^{-1} - \frac{8}{z}(1+3/z)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - 2/z + 4/z^2 - \cdots\right) - \frac{8}{z}\left(1 - 3/z + 9/z^2 - \cdots\right)$$

$$\therefore f(z) = 1 + (3-8)z^{-1} + (24-6)z^{-2} + \cdots = 1 - 5z^{-1} + 18z^{-2} + (-60)z^{-3} + \cdots$$

Which is a Laurent Series.

Eg4. Obtain the Taylor expansion of $f(z) = \frac{1-z}{z^2}$ in powers of $(z-1)$

Soln. We have $f(z) = \frac{1-z}{z^2}$

$f(z)$ is not analytic at $z=0$.
Let us draw a circle with centre 1 and radius less than 1 so that $z=0$ can be excluded. i.e. $|z-1|<1$
Hence the region of validity of Taylor's Series is $|z-1|<1$ ($|z-1|<1$ converges)

$\because f(z) = \frac{1-z}{z^2} = \frac{1}{z^2} - \frac{1}{z} \Rightarrow f(1) = 0$

$f'(z) = -\frac{2}{z^3} + \frac{1}{z^2} \Rightarrow f'(1) = -1$

$f''(z) = +\frac{6}{z^4} - \frac{2}{z^3} \Rightarrow f''(1) = 4$

Taylor's Series in powers of $(z-1)$ can be written as (i.e. about $z=1$)

$$f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \cdots$$

or $f(z) = -(z-1) + 2(z-1)^2 + \cdots = \sum_{n=1}^{\infty} (-1)^n n (z-1)^n$ Ans.

Eg5. Expand $\cos z$ in a Taylor's Series about $z = \pi/4$

Soln. Here $f(z) = \cos z$

Taylor's Series about $z = \pi/4$ will be

$$f(z) = f(\pi/4) + (z-\pi/4)f'(\pi/4) + \frac{(z-\pi/4)^2}{2!}f''(\pi/4) + \cdots$$

$$\therefore \cos z = \frac{1}{\sqrt{2}} + (z-\pi/4)\left(-\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(-1/\sqrt{2}\right) + \cdots$$

$$\cos z = \frac{1}{\sqrt{2}}\left[1 - (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} + \cdots\right]$$ Ans