Fourier transform of the derivatives of a function

Introduction: Problems in which one of the variables ranges from $-\infty$ to ∞ or 0 to ∞, such problems are solved by taking infinite Fourier transform both sides

The fourier transform of the function $u(x,t)$ is $F[u(x,t)]$

i.e. $F[u(x,t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \, e^{isx} dx.$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[\left. e^{isx} \frac{\partial u}{\partial x} \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} is\, e^{isx} \frac{\partial u}{\partial x} dx \right]$$

If u and $\frac{\partial u}{\partial x}$ tend to zero as $x \to \pm\infty$ then

$$\boxed{F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F[u]} \rightarrow \text{fourier transform of derivative}$$

Illy

$$\boxed{F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}} \left\{ s(u)_{x=0} \right\} - s^2 F_s[u]} \rightarrow \text{fourier sine transform of derivative}$$

$$\boxed{F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2 F_c[u]} \rightarrow \text{fourier Cosine transform of derivative}$$

Note. 1. For the exclusion of $\frac{\partial^2 u}{\partial x^2}$ from differential equation we require $(u)_{x=0}$ in sine transformation and $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ in Cosine transform

NB. 2 $(u)_{x=0} = 0$ means $u = 0$ at $x = 0$

$u_x(0,t) = 0$ means $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$ i.e $\frac{\partial u}{\partial x} = 0$ at $x = 0$

Eg 1. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$

if $u(0,t) = 0$, $u(x,0) = e^{-x}$ $(x>0)$

$u(x,t)$ is bounded where $x>0$, $t>0$

Soln We have $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ ... ①

$\because$ $(u)_{x=0}$ is given, we shall apply fourier sine transform both sides in ①

i.e. $F_s\left\{\frac{\partial u}{\partial t}\right\} = F_s\left\{2\frac{\partial^2 u}{\partial x^2}\right\}$

$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = 2\left[\sqrt{\frac{2}{\pi}}\left\{s(u)_{x=0}\right\} - s^2\bar{u}_s\right]$

$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin sx\,dx\right] = -2s^2\bar{u}_s$

$\left[\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx\right]$

$\Rightarrow \frac{d}{dt}\bar{u}_s = -2s^2\bar{u}_s \Rightarrow D\bar{u}_s + 2s^2\bar{u}_s = 0$

$\Rightarrow (D+2s^2)\bar{u}_s = 0 \xRightarrow{AE} D+2s^2 = 0 \Rightarrow D = -2s^2$

$\therefore \bar{u}_s = c_1 e^{-2s^2t}$ ...②

Now $\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx$ $\left(u(x,0) = e^{-x} \text{ given}\right)$

$\Rightarrow \bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\frac{s}{1^2+s^2} = c_1$ from ②

$\boxed{\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}}$

$\therefore$ eqn. ② reduces to

$\bar{u}_s = \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}e^{-2s^2t}$ ...③

Now applying Inverse fourier sine transform both sides in ③

$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_s \sin sx\,ds$

$\therefore u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}e^{-2s^2t}\sin sx\,ds$ Ans.

e.g. 2.
Solve the equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, for $0<x<\infty, t>0$
subject to the condition $u_x(0,t)=0$, $u(x,0)=e^{-ax}$ for $0<x<\infty$

Soln $\because u_x$ (i.e $\left(\frac{\partial u}{\partial x}\right)_{x=0}$) is given
$\therefore$ we shall apply Fourier cosine transform to both side of the given equation.
Given equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ....(1)

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = F_c\left\{c^2 \frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos sx\, dx = c^2 \left[\sqrt{\frac{2}{\pi}} \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 \bar{u}_c\right]$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos sx\, dx\right] = -c^2 s^2 \bar{u}_c$$

$$\Rightarrow \frac{d}{dt} \bar{u}_c + c^2 s^2 \bar{u}_c = 0 \Rightarrow (D + c^2 s^2)\bar{u}_c = 0$$

$$\boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx}$$

$\therefore$ AE, $D + c^2 s^2 = 0 \Rightarrow D = -c^2 s^2$

$\therefore \bar{u}_c(s,t) = C_1 e^{-c^2 s^2 t}$ ....(2)

Now $\bar{u}_c(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \cos sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$ ($u(x,0) = e^{-ax}$ given)

$$\Rightarrow \bar{u}_c(s,0) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} = C_1 \text{ from (2)}$$

$$\boxed{\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}}$$

$\therefore$ eqn. (2) reduces to

$$\bar{u}_c(s,t) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} e^{-c^2 s^2 t} \quad ....(3)$$

Taking Inverse Fourier Cosine transform both sides in (3)

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_c \cos sx\, ds$$

$$\therefore u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+s^2} e^{-c^2 s^2 t} \cos sx\, ds \quad \text{Ans}$$

eg 3 Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x>0$, $t>0$
subject to the conditions (i) $u=0$ when $x=0$, $t>0$
(ii) $u = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1 \end{cases}$ when $t=0$
and (iii) $u(x,t)$ is bounded.

Soln We have $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(1)
∵ the value of u is given at $x=0$, then we shall apply fourier sine transform both sides in eqn (1)

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

or $\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$

$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx\right] = \sqrt{\frac{2}{\pi}}\, s(u)_{x=0} - s^2 \bar{u}_s$

$\Rightarrow \frac{d\bar{u}_s}{dt} + s^2 \bar{u}_s = 0$ by condition (i)

$\Rightarrow (D+s^2)\bar{u}_s = 0 \Rightarrow$ A.E, $D + s^2 = 0 \Rightarrow D = -s^2$

∴ $\bar{u}_s = c_1 e^{-s^2 t}$ ...(2)

when $t=0$, the fourier sine transform of $u(x,t)$ is

Now $\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^1 1 \cdot \sin sx \, dx + \sqrt{\frac{2}{\pi}} \int_1^\infty 0 \cdot \sin sx \, dx$

$\Rightarrow \bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}} \left| -\frac{\cos sx}{s} \right|_0^1 = \sqrt{\frac{2}{\pi}} \left\{\frac{1}{s} - \frac{\cos s}{s}\right\} \Rightarrow \bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}} \left\{\frac{1-\cos s}{s}\right\}$

from (2) $\bar{u}_s(s,0) = c_1 \Rightarrow c_1 = \sqrt{\frac{2}{\pi}} \left\{\frac{1-\cos s}{s}\right\}$

∴ eqn (2) reduces to $\bar{u}_s(s,t) = \sqrt{\frac{2}{\pi}} \frac{(1-\cos s)}{s} e^{-s^2 t}$ ...(3)

Now applying Inverse fourier sine transform both sides in (3)

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_s \sin sx \, ds$$

∴ $u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{1-\cos s}{s} e^{-s^2 t} \sin sx \, ds$ Ans.

Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for $x \geq 0, t \geq 0$ under the given conditions $u = u_0$ at $x=0$, $t>0$ with initial condition $u(x,0)=0$, $x \geq 0$

Soln We have $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ ...(1)

$\because$ the value of u at $x=0$ is given so we will apply Fourier sine transform both sides of (1)

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = k\left[\sqrt{\frac{2}{\pi}}\, s(u)_{x=0} - s^2\bar{u}_s\right]$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin sx\,dx\right] = \sqrt{\frac{2}{\pi}}\,ksu_0 - ks^2\bar{u}_s$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = \sqrt{\frac{2}{\pi}}\,ku_0 - ks^2\bar{u}_s \Rightarrow \frac{d\bar{u}_s}{dt} + ks^2\bar{u}_s = \sqrt{\frac{2}{\pi}}\,ksu_0 \rightarrow (2)$$

Eqn (2) is the linear eqn. linear in $\bar{u}_s$

$$\frac{dy}{dx} + Py = Q$$
$$y \cdot IF = \int (Q \cdot IF)\,dx + c$$
$$IF = e^{\int P\,dx}$$

Now

$$\bar{u}_s e^{ks^2t} = \sqrt{\frac{2}{\pi}}\,ku_0\int s e^{ks^2t}\,dt = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{s}\,e^{ks^2t} + c \quad (3)$$

$$\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin sx\,dx = 0 \quad \because u(x,0)=0$$

$\therefore$ from (3) $\quad 0 = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{s} + c \Rightarrow c = -\sqrt{\frac{2}{\pi}}\,\frac{u_0}{s}$

$\therefore$ eqn (3) reduces to

$$\bar{u}_s e^{ks^2t} = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{s}\left(e^{ks^2t} - 1\right)$$

$$\therefore \bar{u}_s = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{s}\left(1 - e^{-ks^2t}\right) \quad \ldots (4)$$

Now applying Inverse Fourier sine transform both sides of eqn (4)

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_s \sin sx\,ds$$

$$\therefore u(x,t) = \frac{2}{\pi}u_0\int_0^\infty \frac{1}{s}\left(1 - e^{-ks^2t}\right)\sin sx\,ds \quad \text{Ans}$$