## Properties of fourier transforms:-

1) **Linear property**: If $F(s)$ and $G(s)$ are fourier transforms of $f(x)$ and $g(x)$ respectively then

$$F[af(x)+bg(x)] = aF(s)+bG(s),$$ where $a, b$ are constants.

2) **Change of scale property**: If $F(s)$ is the complex fourier transform of $f(x)$ then $F\{f(ax)\} = \frac{1}{a}F(s/a)$, $a \neq 0$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(ax)\,dx$$

putting $ax = t$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist/a} f(t)\,dt \cdot \frac{1}{a} = \frac{1}{a}\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)\,dt$$

$$= \frac{1}{a}F(s/a)$$

similarly $F_s\{f(ax)\} = \frac{1}{a}F_s(s/a)$ and $F_c\{f(ax)\} = \frac{1}{a}F_c(s/a)$

3) **Shifting Property**: If $F(s)$ is the complex fourier transform of $f(x)$, then $F\{f(x-a)\} = e^{isa}F(s)$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{isx}\,dx.$$

$\boxed{\text{Putting } x-a=t}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(t+a)} f(t)\,dt = e^{isa}\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} f(t)\,dt = e^{isa}F(s)$$

4) **Modulation Theorem**: If $F(s)$ is the complex fourier transform of $f(x)$ then $F\{f(x)\cos ax\} = \frac{1}{2}[F(s+a)+F(s-a)]$

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,\frac{e^{iax}+e^{-iax}}{2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2}\left\{\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\,dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx\right\}\right] = \frac{1}{2}[F(s+a)+F(s-a)]$$

(i) $F_s\{f(x)\cos ax\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\times\frac{1}{2}\left[\int_0^{\infty} f(x)\sin(s+a)x\,dx + \int_0^{\infty} f(x)\sin(s-a)x\,dx\right]$$

$\therefore$ $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a)+F_s(s-a)]$

lly (ii) $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(iii) $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

**e.g.** Find the Fourier transform of $e^{-x^2}$. Hence find the Fourier transform of (i) $e^{-ax^2}$, $(a>0)$, (ii) $e^{-x^2/2}$, (iii) $e^{-x^2}\cos 2x$ (iv) $e^{-4(x-3)^2}$

**Sol:** $F\{e^{-x^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2-isx)}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left\{\left(x-\frac{is}{2}\right)^2+\frac{s^2}{4}\right\}} = \frac{1}{\sqrt{2\pi}} e^{-(s^2/4)}\int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2}\,dx$

$= \frac{1}{\sqrt{2\pi}} e^{-(s^2/4)}\int_{-\infty}^{\infty} e^{-t^2}\,dt = e^{-(s^2/4)}\times\frac{1}{\sqrt{2\pi}}\sqrt{\pi}$ $\quad\left\{\because \text{putting } x-\frac{is}{2}=t,\ \int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}\right.$

$\therefore F\{e^{-x^2}\} = \frac{1}{\sqrt{2}} e^{-s^2/4}$ Imp result.

(i) $F(e^{-ax^2}) = F\left(e^{-(\sqrt{a}x)^2}\right) = \frac{1}{\sqrt{a}}\frac{1}{\sqrt{2}} e^{-\frac{1}{4}(s/\sqrt{a})^2}$. (using change of scale prop $F(f(ax)) = \frac{1}{a}F(s/a)$)

$\therefore F(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-(s^2/4a)} \longrightarrow$ Imp. result.

(ii) $F(e^{-x^2/2}) = e^{-s^2/2} \xrightarrow{[\text{Putting } a=1/2]}$ Imp. result. [$e^{-x^2/2}$ and $e^{-s^2/2}$ same function]

(iii) $F(e^{-x^2}\cos 2x) = \frac{1}{2}\times\frac{1}{\sqrt{2}}\left[e^{-\frac{1}{4}(s+2)^2}+e^{-\frac{1}{4}(s-2)^2}\right]$ $\quad\boxed{\because F\{f(x)\cos ax\} = \tfrac{1}{2}[F(s+a)+F(s-a)] \text{ Modulation theorem}}$

[Putting $a=1$ in (i)]

(iv) $F(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-(s^2/4a)} = \frac{1}{\sqrt{2a}} e^{-\frac{1}{4}(s/\sqrt{a})^2}$ from (i)

$F(e^{-4x^2}) = \frac{1}{\sqrt{2}}\frac{1}{2} e^{-1/4(s/2)^2} = \frac{1}{2\sqrt{2}} e^{-s^2/16}$

$F\left(e^{-4(x-3)^2}\right) = \frac{1}{2\sqrt{2}} e^{i3s}\times e^{-s^2/16}$ $\quad\left[\text{using Shifting prop } F\{f(x-a)\} = e^{isa}F(s)\right]$

or $F\left(e^{-4(x-3)^2}\right) = \frac{1}{2\sqrt{2}} e^{\left(i3s-\frac{s^2}{16}\right)}$

Convolution: defn. The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty,\infty)$ is defined as

$$f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du = h(x)$$

## CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

Theorem: The fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their fourier transform.

i.e. $F\{f(x)*g(x)\} = F\{f(x)\}\cdot F\{g(x)\}$

Proof:

$$F\{f(x)*g(x)\} = F\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right\}e^{isx}\,dx$$

Changing the order of integration

$$F\{f(x)*g(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,e^{isx}\,dx\right\}du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,e^{is(x-u)}\,d(x-u)\right\}e^{isu}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isu} f(u)\,du\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} g(t)\,dt\right\} \quad \left(\text{putting } x-u=t\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} g(x)\,dx\;\Big\}$$

$$F\{f(x)*g(x)\} = F\{f(x)\}\cdot F\{g(x)\}$$