FOURIER SINE AND COSINE TRANSFORMS

We know the fourier sine Integral of $f(x)$ as

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin st \sin sx \, dt \, ds \qquad (\lambda \text{ is replaced by } s)$$

or $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st \, dt\right] ds$$

If $\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st \, dt = F_s(s) \longrightarrow$ known as fourier sine transform

then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx \, ds \longrightarrow$ known as Inverse fourier Sine transform

Also we know the fourier Cosine Integral of $f(x)$ as

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos st \cos sx \, dt \, ds \qquad (\lambda \text{ is replaced by } s)$$

or $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st \, dt\right] ds$$

If $\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st \, dt = F_c(s) \longrightarrow$ known as fourier Cosine transform

then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx \, ds \longrightarrow$ known as Inverse fourier Cosine transform

e.g. 1. Find the fourier sine transformation of $e^{-|x|}$, hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}dx = \frac{\pi e^{-m}}{2}$, $m>0$

Soln

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx \, f(x)\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx \, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{1^2+s^2}\right) \qquad \left[\because \int_0^\infty e^{-ax}\sin bx = \frac{b}{a^2+b^2},\ a>0\right]$$

This is the reqd fourier sine transform

The inverse Fourier sine transform is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx \, ds = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)\sin sx \, ds$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{s \sin sx}{1+s^2}ds$$

Putting $x = m$

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s \sin sm}{1+s^2}ds \Rightarrow \frac{\pi e^{-m}}{2} = \int_0^\infty \frac{x \sin mx}{1+x^2}dx$$

Eg.2 Find the Fourier Cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$

Soln Fourier Cosine transform $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-2x} + 4e^{-3x}) \cos sx \, dx$

$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-2x} \cos sx \, dx + 4 \int_0^\infty e^{-3x} \cos sx \, dx \right.$

$= \sqrt{\frac{2}{\pi}} \left\{ \frac{2}{s^2+2^2} + 4 \frac{3}{s^2+3^2} \right] = \sqrt{\frac{2}{\pi}} \times 2 \left\{ \frac{1}{s^2+4} + \frac{6}{s^2+9} \right\}$ Ans

Eg.3 Find the inverse Fourier Sine transform of $\frac{1}{s} e^{-as}$

Soln: Inverse Fourier Sine transform

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s} e^{-as} \sin sx \, ds$ — (1)

$\frac{df}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s} e^{-as} \cos sx \times s \, ds = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx \, ds = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$

$f(x) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+x^2} dx = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a} + c$ — (2)

when $x=0$, $f=0$, $c=0$

$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a}$

Putting $x=0$ in (1) & (2)

$f(0) = c$ in (2)

$f(0) = 0$ in (1) $\Rightarrow c = 0$.

Eg.4 Find the cosine transform of the function $f(x)$

if $f(x) = \cos x, \quad 0<x<a$

$= 0, \quad x>a$

Soln

$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx \, dx = \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \int_0^a \left\{ \cos(s+1)x + \cos(1-s)x \, dx \right.$

$= \sqrt{\frac{1}{2\pi}} \left\{ \left| \frac{\sin(1+s)x}{1+s} \right|_0^a + \left| \frac{\sin(1-s)x}{1-s} \right|_0^a \right\}$

$= \sqrt{\frac{1}{2\pi}} \left\{ \frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s} \right\}$ Ans

Q5 Find the fourier Cosine and Sine transform of $f(x)=e^{-ax}$ for $x>0, a>0$

hence deduce the integral $\int_0^\infty \frac{\cos sx}{a^2+s^2}ds$ & $\int_0^\infty \frac{s\sin sx}{a^2+s^2}ds$

Soln

Fourier Cosine transform

$$F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$F_c(s)=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

$$\boxed{\begin{aligned}\int_0^\infty e^{-ax}\cos bx\,dx &= \frac{a}{a^2+b^2}\\ \int_0^\infty e^{-ax}\sin bx\,dx &= b/(a^2+b^2)\end{aligned}}$$

Now inverse Fourier Cosine transform.

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,F_c(s)ds=\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cos sx\,ds$$

$$e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cos sx\,ds \Rightarrow \int_0^\infty \frac{\cos sx}{a^2+s^2}ds=\frac{\pi e^{-ax}}{2a}\ \text{Ans.}$$

Fourier Sine transform

$$F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx\ \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}$$

$$F_s(s)=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$

Inverse fourier Sine transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\,F_s(s)\,dx=\frac{2}{\pi}\int_0^\infty \frac{s}{a^2+s^2}\sin sx\,ds$$

$$\therefore e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{s}{a^2+s^2}\sin sx\,ds \Rightarrow \int_0^\infty \frac{s\sin sx}{a^2+s^2}ds=\frac{\pi e^{-ax}}{2}$$

$$\int_0^\infty \frac{s\,ds}{a^2+s^2}=\frac{\pi e^{-ax}}{2}.\ \text{Ans}$$

eg 6 Find fourier sine tranform of $\frac{e^{-ax}}{x}$

Soln Let $f(x) = \frac{e^{-ax}}{x}$

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, f(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, \frac{e^{-ax}}{x}\,dx$$

differentiating both sides w.r.to s

$$\frac{d}{ds}\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

or $\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\, a \int \frac{ds}{s^2+a^2} = \frac{a}{a}\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) + c$

$\therefore \bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$ (when $s=0$, $\bar{f}(s) = 0 \Rightarrow c = 0$

$F_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}(s/a)$ is the reqd Fourier sine transform.

e.g 7 Find fourier Cosine transform of $f(x) = \frac{1}{1+x^2}$, and hence derive fourier sine transform $\phi(x) = \frac{x}{1+x^2}$

Soln $F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos sx\,dx$ ... (1)

$$\frac{dF_c}{ds} = \sqrt{\frac{2}{\pi}}\int_0^\infty -\frac{x\sin sx}{(1+x^2)}\,dx = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+x^2-1)\sin sx}{x(1+x^2)}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$\frac{dF_c}{ds} = -\sqrt{\frac{2}{\pi}}\times\frac{\pi}{2} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \ldots (2)$$

$$\frac{d^2F_c}{ds^2} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{(1+x^2)}\,dx = F_c(s) \quad \text{from (1)}$$

$$\frac{d^2F_c}{ds^2} - F_c = 0 \Rightarrow \left(\frac{d^2}{ds^2}-1\right)F_c = 0 \Rightarrow (D^2-1)F_c = 0 \Rightarrow D^2-1=0 \Rightarrow D = \pm 1$$

$\therefore F_c(s) = c_1e^{s} + c_2e^{-s}$ ... (3)

Putting $s=0$ $\quad c_1+c_2 = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}dx$ from (1)

$$c_1+c_2 = \sqrt{\frac{2}{\pi}}\left|\tan^{-1}x\right|_0^\infty = \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

diff eqn (3) $\quad \frac{dF_c}{ds} = c_1e^{s} - c_2e^{-s}$

for $s=0$, $\quad c_1 - c_2 = -\sqrt{\pi/2}$ from (2)

hence $c_1 = 0$, $c_2 = \sqrt{\pi/2}$ $\quad \therefore$ from (3) $F_c(s) = \sqrt{\frac{\pi}{2}}e^{-s}$. reqd $F_c(s)$

$$\frac{dF_c}{ds} = -\sqrt{\frac{\pi}{2}}e^{-s} = \sqrt{\frac{2}{\pi}}\int_0^\infty -\frac{x\sin sx}{1+x^2}\,dx,\ \text{Now } F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = \sqrt{\frac{\pi}{2}}e^{-s}$$

e.g.8. Find the Fourier sine and cosine transform of $F(x) = x$

Soln. $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \cos sx \, dx$

and $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \sin sx \, dx$.

$F_c(s) - i F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x(\cos sx - i \sin sx)\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x e^{-isx} dx$.

Putting $isx = y$

$\therefore F_c(s) - i F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{y}{is} e^{-y} \frac{dy}{is}$

$= -\sqrt{\frac{2}{\pi}} \frac{1}{s^2} \int_0^{\infty} y^{2-1} e^{-y} dy \qquad \left[\because \int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma n\right]$

$= -\sqrt{\frac{2}{\pi}} \frac{1}{s^2} \Gamma 2 = -\sqrt{\frac{2}{\pi}} \frac{1}{s^2}$

$\therefore F_c(s) - i F_s(s) = -\sqrt{\frac{2}{\pi}} \frac{1}{s^2} - i \cdot 0$

$\Rightarrow F_c(s) = -\sqrt{\frac{2}{\pi}} \frac{1}{s^2}$ and $F_s(s) = 0$