FOURIER TRANSFORM

We know the Fourier Integral of $f(x)$ as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

i.e

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\right]d\lambda \quad \ldots (A)$$

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\right]d\lambda \quad \ldots (B)$$

$\because \cos\lambda(t-x)$ is an even function of λ, $f(t)$ does not depend on λ.

also

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\sin\lambda(t-x)\,dt\right]d\lambda = 0 \quad \ldots (C)$$

$\because \sin\lambda(t-x)$ is an odd fn of λ.

$\therefore$ eqn (B) + i eqn (C)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\,e^{i\lambda(t-x)}\,dt\,d\lambda \quad \ldots (D)$$

This is known as complex form of Fourier Integral.

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\,e^{is(t-x)}\,dt\,ds \quad \ldots (D')$$

replacing λ by s.

writing the exponential function as a product of exponential functions

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)\,e^{ist}\,dt\right] e^{-isx}\,ds$$

If

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)\,e^{ist}\,dt = F(s) \rightarrow \text{This is called fourier transform}$$

then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(s)\,e^{-isx}\,ds \rightarrow \text{This is called Inverse fourier transform.}$$

e.g 1. Find the fourier transform of the function

$$f(x) = \begin{cases} 1+\frac{x}{a}, & -a<x<0 \\ 1-\frac{x}{a}, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$$

Soln fourier transform $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)\,e^{isx}\,dx$

or

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^0 \left(1+\frac{x}{a}\right)e^{isx}\,dx + \int_0^a \left(1-\frac{x}{a}\right)e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left|\left(1+\frac{x}{a}\right)\frac{e^{isx}}{is} - \frac{1}{a}\frac{e^{isx}}{(is)^2}\right|_{-a}^0 + \left|\left(1-\frac{x}{a}\right)\frac{e^{isx}}{is} + \frac{1}{a}\frac{e^{isx}}{(is)^2}\right|_0^a\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is} + \frac{1}{as^2} - \frac{e^{-isa}}{as^2} + 0 - \frac{1}{is} - \frac{e^{isa}}{as^2} + \frac{1}{as^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{1}{as^2}\left(e^{isa} + e^{-isa}\right)\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{1}{as^2} \times 2\cos as\right]$$

$$= \frac{2}{\sqrt{2\pi}}\cdot\frac{1}{as^2}\left[1-\cos as\right] = \frac{2\times 2}{\sqrt{2\pi}}\frac{1}{as^2}\sin^2\frac{as}{2} = \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{1}{as^2}\sin^2\frac{as}{2}$$

$\therefore F(s) = \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{1}{as^2}\cdot\sin^2\frac{as}{2}$ is the reqd fourier transform, Provided $s \neq 0$

ex2 Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

hence evaluate (a) $\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} ds$ (b) $\int_0^{\infty} \frac{\sin s}{s} ds$

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} e^{isx} f(x)\,dx + \int_{-a}^{a} e^{isx} f(x)\,dx + \int_{a}^{\infty} e^{isx} f(x)\,dx \right]$$

Putting $x = -y$ in the 1st integral

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{\infty}^{a} e^{-isy} f(-y)\,dy + \int_{-a}^{a} e^{isx} f(x)\,dx + 0 \right] = \frac{1}{\sqrt{2\pi}} \left[\int_{a}^{\infty} e^{-isy} f(-y)\,dy + \int_{-a}^{a} e^{isx} f(x)\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-a}^{a} e^{isx}\,dx \right] = \frac{1}{\sqrt{2\pi}} \left| \frac{e^{isx}}{s} \right|_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isa} - e^{-isa}}{is} \right)$$

$$\boxed{F(s) = \frac{1}{\sqrt{2\pi}} \, \frac{2 \sin as}{s}, \quad s \ne 0,}$$

If $s = 0$. $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dx = \frac{1}{\sqrt{2\pi}} \times 2a.$

Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\,ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \times \frac{1}{\sqrt{2\pi}} \cdot 2 \frac{\sin as}{s}\,ds.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-isx} \frac{\sin as}{s} ds = \frac{1}{\pi} \int_{-\infty}^{\infty} (\cos sx - i \sin sx) \frac{\sin as}{s} ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \sin as}{s} ds$$ (2nd part is odd fn hence zero)

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \sin as}{s} ds = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{\cos sx \sin as}{s} ds = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a. \end{cases}$$

(b) $$2 \int_0^{\infty} \frac{\cos sx \sin as}{s} ds = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

Putting $x = 0$ & $a = 1$

$$\int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

eg 2 Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$

$\cos s + i\sin s$
$+\cos x - i\sin s$
$\cos s + i\sin s$
$-\cos s + i\sin s$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-1} f(x) e^{isx} dx + \int_{-1}^{1} f(x) e^{isx} dx + \int_1^\infty f(x) e^{isx} dx \right]$$

Putting $x = -y$ in the first integral

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\int_{\infty}^{1} f(-y) e^{-isy} dy + \int_{-1}^{1} (1-x^2) e^{isx} dx + 0 \right] = \frac{1}{\sqrt{2\pi}} \left\{ \int_1^\infty f(-y) e^{-isy} dy + \int_{-1}^{1} (1-x^2) e^{isx} dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{1} (1-x^2) e^{isx} dx \right\} = \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{(1-x^2)e^{isx}}{is}\right]_{-1}^{1} + 2\int_{-1}^{1} \frac{x e^{isx}}{is} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \left\{ \left|\frac{x e^{isx}}{(is)^2}\right|_{-1}^{1} - \left|\frac{e^{isx}}{(is)^3}\right|_{-1}^{1} \right\} = \frac{2}{\sqrt{2\pi}(is)^2} \left\{ (e^{is} + e^{-is}) - \left(\frac{e^{is} - e^{-is}}{is}\right) \right\}$$

$$= \frac{-2}{\sqrt{2\pi}\, s^2} \left\{ 2\cos s - \frac{2\sin s}{s} \right\} = -\frac{4}{\sqrt{2\pi}\, s^2} \left\{ \frac{s\cos s - \sin s}{s} \right\} = \frac{-4}{\sqrt{2\pi}} \left(\frac{s\cos s - \sin s}{s^3} \right)$$

$$\Rightarrow F(s) = -\frac{4}{\sqrt{2\pi}} \left(\frac{s\cos s - \sin s}{s^3} \right)$$

Inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds = -\frac{4}{2\pi} \int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) (\cos sx - i\sin sx)\, ds$$

$$= -\frac{2\cdot 2}{\pi} \left\{ \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3} \right) \cos sx\, ds = -\frac{4}{\pi} \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3} \right) \cos sx\, ds \right.$$

Putting $x = 1/2$:

$$-\frac{4}{\pi} \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3} \right) \cos\frac{s}{2}\, ds = \begin{cases} 1 - \frac{1}{4}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$\int_0^\infty \left(\frac{s\cos s - \sin s}{s^3} \right) \cos\frac{s}{2}\, ds = -\frac{3\pi}{16}$$

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx = -\frac{3\pi}{16}$$