Complex form of Fourier Series

The complex form of Fourier Series is useful in some electrical circuits and in the digital signal processing. This is especially useful in problems on electrical circuits having impressed periodic voltage.

The Fourier series of a periodic function f(x) of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \dots (i)$$

we know that $\cos\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$ $\quad \sin\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$

from (i)

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l}+e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l}-e^{-in\pi x/l}}{2i}\right)\right\}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left\{\frac{1}{2}(a_n - ib_n)e^{in\pi x/l}\right\} + \left\{\frac{1}{2}(a_n+ib_n)e^{-in\pi x/l}\right\}\right]$$

Taking $\frac{a_0}{2} = c_0$; $\frac{1}{2}(a_n - ib_n) = c_n$ and $\frac{1}{2}(a_n+ib_n) = c_{-n}$

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{in\pi x/l} + c_{-n}e^{-in\pi x/l}\right\} \quad \dots (ii)$$

Now $c_n = \frac{1}{2l}\left[\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx$$

$$\therefore c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx \quad (iii)$$

Illy $c_{-n} = \frac{1}{2l}\left[\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx\right.$

$$\therefore c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}dx \quad \dots (iv)$$

combining these, we have $\quad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx \quad \dots (v)$

where $n = 0, \pm 1, \pm 2, \pm 3 \dots$

Then the series (ii) can be compactly written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \quad \dots \text{(CF-1)}$$

This is known as complex form of fourier series

In general $\quad c_n = \frac{1}{2l}\int_{\alpha}^{\alpha+2l} f(x)e^{-in\pi x/l}, \quad n = 0, \pm 1, \pm 2, \dots,$ (interval α to $\alpha+2l$)

e.g.-1 Find the complex form of Fourier series of $f(x) = e^{2x}$, $0<x<2$

Soln Here $l = 1$

The complex form of Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$

i.e. $e^{2x} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$ .. (1)

where $c_n = \frac{1}{2}\int_0^2 e^{2x} e^{-in\pi x}\,dx = \frac{1}{2}\left[\left|\frac{e^{(2-in\pi)x}}{2-in\pi}\right|_0^2\right]$

$= \frac{1}{2(2-in\pi)}\left(e^{(2-in\pi)2}-1\right) = \frac{1}{4-2in\pi}(e^4-1)$ $(\because e^{-2in\pi} = 1$

$= (e^4-1)\left(\frac{4+2in\pi}{16+4n^2\pi^2}\right)$

hence $e^{2x} = (e^4-1)\sum_{n=-\infty}^{\infty}\left(\frac{4+2in\pi}{16+4n^2\pi^2}\right)e^{in\pi x}$ Ans

e.g. 2 Find the complex form of Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$

Soln The complex form of Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$

i.e. $e^{-x} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$

when $c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\,dx$

$\because l = 1$

$c_n = \frac{1}{2}\left[\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1}\right] = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$

$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$

$= \frac{(e-e^{-1})}{2}\frac{(-1)^n(1-in\pi)}{1+n^2\pi^2} = (-1)^n\frac{(1-in\pi)}{1+n^2\pi^2}\sinh 1$

hence $e^{-x} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1 \cdot e^{in\pi x}$ Ans

H.A Find the complex Fourier series of $f(x) = e^{x}$ if $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$