Parseval's formula

1. $\int_{-l}^{l} [f(x)]^2 dx = l\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2+b_n^2) \right\}$

2. $\int_0^{2l} [f(x)]^2 dx = l\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2+b_n^2) \right\}$

3. half range cosine series $(0,l)$

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots \infty\right)$$

4. half range sine series $(0,l)$

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left(b_1^2 + b_2^2 + b_3^2 + \cdots\right)$$

eg By using the sine series for $f(x)=1$, in $0<x<\pi$ show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Soln $f(x) = \Sigma b_n \sin nx$ half range sine series

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} 1.\sin nx\,dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}$

$= -\frac{2}{n\pi}\left((-1)^n - 1\right) \Rightarrow b_n = \frac{4}{n\pi}$ if n odd & $b_n = 0$ if n is even.

$$\int_0^{\pi} [f(x)]^2 dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + \cdots\right]$$

$$\int_0^{\pi} dx = \frac{\pi}{2} \times \frac{16}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\pi = \frac{8}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right].$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

eg 2. Prove that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \cdots\right],$$

hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

<u>Soln</u> Half Range Cosine Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where } f(x) = x(\pi - x),\ a_0 = \frac{2}{\pi}\int_0^{\pi} (x(\pi - x))\,dx.$$

$$\& \ a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx = \frac{2}{\pi}\left[\left|\frac{\pi x^2}{2}\right|_0^{\pi} - \left|\frac{x^3}{3}\right|_0^{\pi}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx = \frac{2}{\pi}\left[\left|(\pi x - x)\frac{\sin nx}{n}\right|_0^{\pi} - \int_0^{\pi}(\pi - 2x)\frac{\sin nx}{n}\,dx\right]$$

$$= +\frac{2}{\pi}\left[\left|(\pi - 2x)\frac{\cos nx}{+n^2}\right|_0^{\pi}\right] = +\frac{2}{\pi}\left|\frac{\pi}{n^2}\{-(-1)^n - 1\}\right| = \frac{2(-2)}{n^2} = \frac{-4}{n^2} \text{ if } n \text{ even}$$

$$= 0 \text{ if } n \text{ odd.}$$

hence $x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \cdots\right]$ Proved

By Parseval formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} (\pi^2 x^2 - 2\pi x^3 + x^4)\,dx = \frac{1}{2}\left(\frac{\pi^2}{3}\right)^2 + 16\left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right)$$

$$\Rightarrow \frac{2}{\pi}\left(\pi^2 \cdot \frac{\pi^3}{3} - 2\pi\frac{\pi^4}{4} + \frac{\pi^5}{5}\right) = \frac{\pi^4}{18} + \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right)$$

$$\Rightarrow \frac{\pi^4}{15} = \frac{\pi^4}{18} + \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) \Rightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ Ans. frs.}$$