V Laplace Transform of Periodic functions

Periodic function: If at equal intervals of abscissa x, the values of each ordinate $f(x)$ repeats itself i.e. $f(x) = f(x+\alpha)\ \forall x \Rightarrow f(x)$ is a periodic function of period α.

Laplace transform of Periodic function: Let $f(t)$ is a periodic function with period T, i.e. $f(t+T) = f(t)$ then

$$\boxed{L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}}$$

Proof: we know $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Putting $t = u+T$ in second integral, $t = u+2T$ in the third integral and so on then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \cdots$$

$$\boxed{\because f(u) = f(u+T) = f(u+2T) = \cdots}$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \cdots$$

$$= (1 + e^{-sT} + e^{-2sT} + \cdots)\int_0^T e^{-st} f(t)\,dt$$

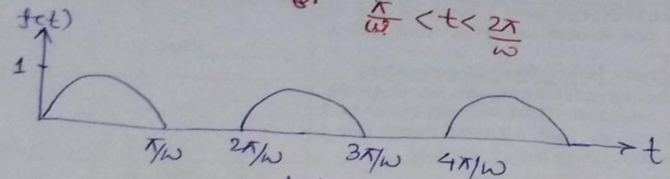
$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt \qquad \left[\because 1 + a + a^2 + \cdots = \frac{1}{1-a}\right]$$

Hence proved.

eg. 1 Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin\omega t, & \text{for } 0 < t < \frac{\pi}{\omega} \\ 0, & \text{for } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Soln



Half wave rectifier

$f(t)$ is a periodic function with $2\pi/\omega$ period

$$L\{f(t)\} = \frac{1}{1-e^{-s\frac{2\pi}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt + 0$$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}} \left| \frac{e^{-st}\left(-s\sin\omega t - \omega\cos\omega t\right)}{s^2+\omega^2} \right|_0^{\omega}$$

$$\left\{ \int e^{ax}\sin bx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \right.$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-\frac{\pi s}{\omega}}(\omega) + \omega}{s^2+\omega^2} \right]$$

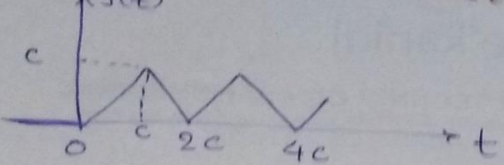
$$= \frac{\omega\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)\left(1+e^{-\pi s/\omega}\right)\left(1-e^{-\pi s/\omega}\right)}$$

$$= \frac{\omega}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)} \quad \text{Ans}$$

Q.2 Find the Laplace transform of the triangular wave function of period 2c given by

$$f(t) = \begin{cases} t, & 0<t<c \\ 2c-t, & c<t<2c \end{cases}$$

Soln



$$L\{f(t)\} = \frac{1}{1-e^{-s2c}} \int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{2cs}} \left[\int_0^c e^{-st} \cdot t\,dt + \int_c^{2c} e^{-st}(2c-t)\,dt \right]$$

$$= \frac{1}{1-e^{2cs}} \left[\left| -\frac{te^{-st}}{s} \right|_0^c - \left| \frac{e^{-st}}{s^2} \right|_0^c + \left| \frac{e^{-st}(2c-t)}{-s} \right|_c^{2c} + \left| \frac{e^{-st}}{s^2} \right|_c^{2c} \right]$$

$$= \frac{1}{1-e^{-2cs}} \left[-\frac{ce^{-st}}{s} - \frac{e^{-sc}}{s^2} + \frac{ce^{-sc}}{s} + \frac{e^{-2sc}}{s^2} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} \right]$$

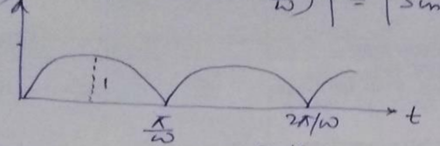
$$= \frac{1}{1-e^{-2cs}} \left(\frac{1 - 2e^{-cs} + e^{-2cs}}{s^2} \right)$$

$$= \frac{1}{s^2} \frac{(1-e^{-cs})^2}{(1+e^{-cs})(1-e^{-cs})} = \frac{1}{s^2} \frac{(1-e^{-cs})}{(1+e^{-cs})} \text{ Ans}$$

eg.3 Find the Laplace transform of the function $f(t) = |\sin \omega t|, \; t \geq 0$

Soln: Let $\left|\sin \omega\left(t+\frac{\pi}{\omega}\right)\right| = |\sin \omega t|$ for any t

$\therefore T = \frac{\pi}{\omega}$



Full wave rectifier

we have

$$L\{f(t)\} = \frac{1}{1-e^{-s\frac{\pi}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin \omega t \, dt\right]$$

$$= \frac{1}{1-e^{-s\pi/\omega}}\left[\left| e^{-st}\left(\frac{-s\sin\omega t - \omega\cos\omega t}{s^2+\omega^2}\right)\right|_0^{\pi/\omega}\right]$$

$$= \frac{1}{1-e^{-s\pi/\omega}} \frac{1}{s^2+\omega^2}\left[e^{-s\pi/\omega}(\omega) + \omega\right]$$

$$= \frac{1}{1-e^{-s\pi/\omega}} \cdot \frac{1}{s^2+\omega^2}\left[\omega\left(1+e^{-s\pi/\omega}\right)\right]$$

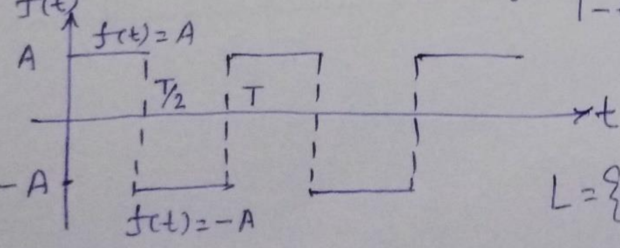
$$= \frac{\omega}{s^2+\omega^2} \; \frac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}}$$

H.A

- Find the Laplace transform of the periodic function (saw tooth wave)

$$f(t) = \frac{kt}{t} \text{ for } 0<t<T, \; f(t+T) = f(t)$$

- Obtain Laplace transform of rectangular wave given by $L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$



$$L = \{f(t)\} = \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} e^{-st} A\,dt + \int_{T/2}^{T} e^{-st}(-A)\,dt\right]$$