IV Unit Step function (or Heaviside's unit function)

When inverse transform cannot be determined from formula we introduce the unit step functions also known as Heaviside's unit function or Heaviside function.

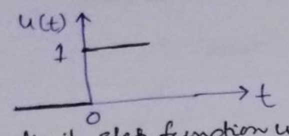
Def: The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t<a \\ 1, & \text{for } t \geq a \end{cases} \quad \text{where } a \geq 0 \quad \cdots (H\text{-}1)$$

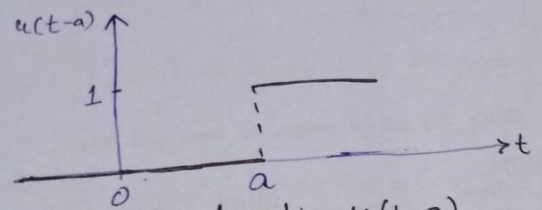
as a particular case $$u(t) = \begin{cases} 0, & \text{for } t<0 \\ 1, & \text{for } t \geq 0 \end{cases} \quad \cdots (H\text{-}2)$$

This is also denoted as $H(t-a)$ or U_a

The product $$f(t)\cdot u(t-a) = \begin{cases} 0, & \text{for } t<a \\ f(t), & \text{for } t \geq a \end{cases} \quad \cdots (H\text{-}3)$$



Unit step function $u(t)$



Unit step function $u(t-a)$

The function $f(t-a)\cdot u(t-a)$ represents the graph of $f(t)$ shifted through a distance 'a' to the right

Laplace transform of Unit Step function:

$$L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

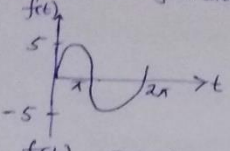
$$= \left| -\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s}$$

$$\therefore \boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}} \quad \cdots (H\text{-}4)$$

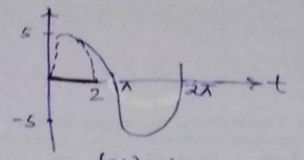
In particular $L\{u(t)\} = \frac{1}{s}$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on". Multiplying function $f(t)$ with $u(t-a)$, we can produce all sorts of effects.

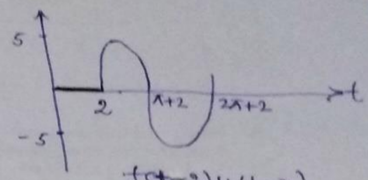
eg Given function $f(t) = 5\sin t$



$f(t) = 5\sin t$



$f(t)u(t-2)$
switched off between $t=0$ and $t=2$ (because $u(t-2)=0$ when $t<2$ and is switched on beginning at $t=2$



$f(t-2)u(t-2)$
It is shifted to the right by 2 seconds, so that it begins 2 seconds later in the same fashion as before.

Second shifting property: If $L\{f(t)\} = \bar{f}(s)$ then
$$L\{f(t-a)\cdot u(t-a)\} = e^{-as}\bar{f}(s)$$

Proof: $L\{f(t-a)\cdot u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$

$$= \int_0^a e^{-st} f(t-a)\cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

Putting $t-a=u$.

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du = e^{-sa}\int_0^\infty e^{-su} f(u)\, du = e^{-sa}\bar{f}(s)$$

e.g. Using unit step function, find the Laplace transform of

$$f(t) \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \geqslant 2\pi \end{cases}$$

Soln

$$f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,[u(t-2\pi)]$$

$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

$$L\{f(t)\} = L\{\sin t\} + L[\{\sin 2t - \sin t\} \cdot u(t-\pi)] + L[\{\sin 3t - \sin 2t\} \cdot u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\bar{f}(s) + e^{-2\pi s}\bar{f}(s)$$

$\because L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{f(t-a) \cdot u(t-a)\} = e^{-as}\bar{f}(s)$

$$L\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+2^2} - \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+3^2} - \frac{2}{s^2+2^2}\right] \text{ Ans.}$$

e.g Using unit step function, find the inverse Laplace transform of $\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

Soln

$$L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a) \cdot u(t-a)$$

$$L^{-1}\left\{\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2} \cdot \frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s} \cdot \frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\tfrac{1}{2}\right) \cdot u\left(t-\tfrac{1}{2}\right) + \sin\pi(t-1) \cdot u(t-1)$$

$$= \cos\pi t \cdot u\left(t-\tfrac{1}{2}\right) + \sin\pi t \cdot u(t-1) \text{ Ans}$$

$\left[\because L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos\pi t\right.$

$\left.L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = \sin\pi t\right]$

V. Unit Impulse Function (or Dirac delta function) or Short Impulses

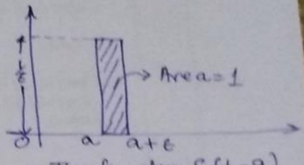
Phenomena of an impulsive nature, such as the action of very large forces (or voltages) over very short intervals of time. For instance: when a tennis ball is hit, a system is given a blow by a hammer, an airplane makes a "hard landing etc. **lightning** on the sky: short interval with high voltage electricity, etc.

The unit Impulse function is also known as Dirac delta function.

The unit impulse function is considered as the limiting form of the function

$$\delta_\epsilon(t-a) = \begin{cases} 1/\epsilon, & a \le t \le a+\epsilon \\ 0, & \text{otherwise} \end{cases}$$

as $\epsilon \to 0$ the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.



The function $\delta_\epsilon(t-a)$

The unit impulse function $\delta(t-a)$ is defined as:

$$\delta(t-a) = \begin{cases} \infty, & \text{for } t=a \\ 0, & \text{for } t \neq a \end{cases} \quad \cdots (D\text{-}1)$$

such that $\int_0^\infty \delta(t-a)\,dt = 1 \quad (a \geq 0)$ [area of the strip]

Laplace transform of unit impulse function:

Let $f(t)$ be a function of t continuous at $t=a$ then

$$\int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt = \int_0^{a+\epsilon} f(t)\cdot\frac{1}{\epsilon}\,dt$$

$$= (a+\epsilon-a)\,f(\eta)\cdot\frac{1}{\epsilon} = f(\eta) \text{ where } a<\eta<a+\epsilon$$

by Mean value theorem $\int_a^b f(t)\,dt = (b-a)f(\eta)$

as $\epsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a) \quad \cdots (D\text{-}2)$

In particular

If $f(t) = e^{-st}$, we get $L[\delta(t-a)] = e^{-as} \qquad a \geq 0,\ s>0$

$\because \int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$