III Convolution: Defn: Let $f(t)$ and $g(t)$ be two function of class A, then convolution of two functions $f(t)$ and $g(t)$ denoted by $F * G$ is defined by the relation

$$F * G = \int_0^t f(u)\, g(t-u)\, du.$$

Convolution theorem of Laplace Transform

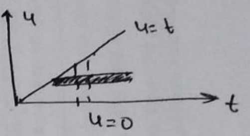
If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$

then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du = F * G \quad \cdots (C-1)$

Proof Let $\phi(t) = \int_0^t f(u) g(t-u)\, du$ —(1)

then $L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)\,dt = \int_0^\infty e^{-st}\,dt \times \int_0^t f(u) g(t-u)\,du$

changing the order of integration



$$= \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, du\, dt$$

$$= \int_0^\infty e^{-su} f(u)\, du \int_u^\infty e^{-s(t-u)} g(t-u)\, dt$$

$$= \int_0^\infty e^{-su} f(u)\, du \int_0^\infty e^{-sv} g(v)\, dv \qquad t-u=v$$

$$L\{\phi(t)\} = \bar{f}(s)\,\bar{g}(s)$$

$$\therefore L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \phi(t) = \int_0^t f(u)\, g(t-u)\, du$$

hence proved

Properties

(i) $F * G = G * F$, (ii) $(F*G)*H = F*(G*)$

(iii) $F*(G+H) = F*G + F*H$

Eg.1 Apply convolution theorem to evaluate

(i) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ (ii) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

(i) Solution $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\frac{1}{(s^2+a^2)}\right\}$

we know that $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$, $L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\} = \frac{1}{a}\sin at$

We know convolution theorem: $L^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\} = \int_0^t f(u)g(t-u)\,du$

we have $L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1}{a}\sin au\,\cos a(t-u)\,du$

$= \frac{1}{a}\times\frac{1}{2}\int_0^t\left[\sin(au+at-au)+\sin(au-at+au)\right]du$

$= \frac{1}{2a}\int_0^t\left[\sin at+\sin(2au-at)\right]du$

$= \frac{1}{2a}\left[\left|u\sin at\right|_0^t - \left|\frac{\cos(2au-at)}{2a}\right|_0^t\right\}$

$= \frac{1}{2a}\left[t\sin at - \frac{(\cos at-\cos at)}{2a}\right] = \frac{1}{2a}t\sin t$ Ans

(ii) Soln $L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+2^2)}\right\} = \int_0^t f(u)\,g(t-u)\,du$

Here $L^{-1}\{1/s\}=1$, $L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$

$\therefore L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+2^2)}\right\} = \int_0^t 1\,\frac{1}{2}\sin 2(t-u)\,du$

$= \frac{1}{2}\left|-\frac{\cos 2(t-u)}{-2}\right|_0^t = \frac{1}{4}(1-\cos 2t)$ Ans.

e.g. 2. Find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$ using Convolution theorem

Sol^n We know that $L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\} = \frac{1}{a}\sin at$

$\therefore$ $L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \frac{1}{a^2}\int_0^t \sin au \sin a(t-u)\,du$ by Convolution theorem

$= \frac{1}{2a^2}\int_0^t [\cos(au-at+au) - \cos(au+at-au)]\,du$

$= \frac{1}{2a^2}\left[\int_0^t \{\cos(2au-at) - \cos at\}\,du\right]$

$= \frac{1}{2a^2}\left[\left|\frac{\sin(2au-at)}{2a}\right|_0^t - \left|u\cos at\right|_0^t\right]$

$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} + \frac{\sin at}{2a} - t\cos at\right] = \frac{1}{2a^2}\left(\frac{\sin at}{a} - t\cos at\right)$

$= \frac{1}{2a^3}[\sin at - at\cos at]$ Ans.

e.g. 3 Apply the Convolution theorem to evaluate $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Sol^n We know that $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$ $\quad\because L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!}$

and $L^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\quad\because L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

$\therefore$ $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)\,du = \int_0^t (ut-u^2)e^{-u}\,du$

$= \left|-(ut-u^2)e^{-u}\right|_0^t + \int_0^t (t-2u)e^{-u}\,du$

$= 0 + \left|-(t-2u)e^{-u}\right|_0^t + 2\int_0^t -e^{-u}\,du$

$= te^{-t} + t + \left|2e^{-u}\right|_0^t$

$= te^{-t} + t + 2e^{-t} - 2$ Ans.

H.A Apply Convolution theorem to evaluate

(1) $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$ (2) $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$

(3) $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$, (4) $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$