II. Inverse Laplace Transform

Def: If $L\{f(t)\} = \bar{f}(s)$ is the Laplace transform then $L^{-1}\{\bar{f}(s)\} = f(t)$ is called Inverse Laplace transform.

Standard form

(i) $L^{-1}\left\{\frac{1}{s}\right\} = 1$, (ii) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$.

(iii) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3 \ldots$

(iv) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!}$

(v) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$, (vi) $L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = \cos at$

(vii) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$, (viii) $L^{-1}\left\{\frac{s}{(s^2-a^2)}\right\} = \cosh at$

(ix) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$, (x) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

(xi) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$

(xii) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$

e.g. 1 Evaluate $L^{-1}\left\{\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}$

Soln: Given Inverse $= 6L^{-1}\left\{\frac{1}{2s-3}\right\}-3L^{-1}\left\{\frac{1}{9s^2-16}\right\}+4L^{-1}\left\{\frac{s}{9s^2-16}\right\}+8L^{-1}\left\{\frac{1}{16s^2+9}\right\}$
$-6\left\{\frac{s}{16s^2+9}\right\}$

$= 3L^{-1}\left\{\frac{1}{s-3/2}\right\}-\frac{1}{3}L^{-1}\left\{\frac{1}{s^2-(4/3)^2}\right\}-\frac{4}{9}L^{-1}\left\{\frac{s}{s^2-(4/3)^2}\right\}$
$+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+(3/4)^2}\right\}-\frac{3}{8}L^{-1}\left\{\frac{s}{s^2+(3/4)^2}\right\}$

$= 3e^{(3/2)t}-\frac{1}{3}\cdot\frac{3}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{1}{2}\,\frac{4}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$

$= 3e^{\frac{3}{2}t}-\frac{1}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{2}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$ Ans

e.g. 2 Evaluate $L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

Soln: Given Inverse $= L^{-1}\left\{\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)\frac{1}{b^2-a^2}\right\}$

$= \frac{1}{b^2-a^2}\left[L^{-1}\left\{\frac{s}{s^2+a^2}\right\}-L^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]=\frac{1}{b^2-a^2}\left[\cos at-\cos bt\right]$

e.g. 3 Evaluate $L^{-1}\left\{\frac{2s-5}{9s^2-25}\right\}$

Soln: $L^{-1}\left\{\frac{2s-5}{9s^2-25}\right\} = 2L^{-1}\left\{\frac{s}{9s^2-25}\right\}-5L^{-1}\left\{\frac{1}{9s^2-25}\right\}$

$= \frac{2}{9}L^{-1}\left\{\frac{s}{s^2-(5/3)^2}\right\}-\frac{5}{9}L^{-1}\left\{\frac{1}{s^2-(5/3)^2}\right\}$

$= \frac{2}{9}\cosh\frac{5}{3}t-\frac{5}{9}\cdot\frac{3}{5}\sinh\frac{5}{3}t$

$= \frac{2}{9}\cosh\frac{5}{3}t-\frac{1}{3}\sinh\frac{5}{3}t$ Ans e.g.4 Find the inverse Laplace transform of $\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Soln: Let $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

$2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$

$s=-1 \Rightarrow A = -\frac{1}{6}$, $s=2 \Rightarrow B = -\frac{4}{3}$, $s=3 \Rightarrow C = 7/2$

$\therefore L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = -\frac{1}{6}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{7}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$

$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$ Ans $\quad \because L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

e.g.5 Find the inverse Laplace transform of $\left\{\frac{s+1}{s^2+s+1}\right\}$

Soln Let $\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2})+\frac{1}{2}}{(s+\frac{1}{2})^2+3/4} = \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{2}\cdot\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$\therefore L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}\cdot L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+(\sqrt{3}/2)^2}\right\}$

$= e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2})t + \frac{1}{2}\cdot\frac{1}{(\sqrt{3}/2)}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2})t$

$= \frac{1}{\sqrt{3}}e^{-t/2}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right]$

$\because L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$ and $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

H.A Evaluate $L^{-1}\left\{\frac{6s^2+22s+18}{s^3+6s^2+11s+6}\right\}$

First Shifting theorem

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t) = e^{at}L^{-1}\{\bar{f}(s)\}$

e.g.6 Find $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$ using first shifting theorem

Soln $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} = L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+16}\right\}$

using first shifting theorem

$$= e^{-3t}L^{-1}\left\{\frac{s-2}{s^2+16}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{s}{s^2+4^2}\right\} - L^{-1}\left\{\frac{2}{s^2+4^2}\right\}\right]$$

$$= e^{-3t}\left[\cos 4t - \frac{2}{4}\sin 4t\right] = e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right] \text{ Ans}$$

e.g.7 find $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

Soln $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = L^{-1}\left\{\frac{4s+16-4}{(s+4)^2}\right\} = 4L^{-1}\left\{\frac{(s+4)-1}{(s+4)^2}\right\}$

$$= 4e^{-4t}L^{-1}\left\{\frac{s-1}{s^2}\right\}$$ using first shifting th

$$= 4e^{-4t}\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}\right] = 4e^{-4t}[1-t] \text{ Ans}$$

Transform of Integral

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

and $L^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(u)\,du$

e.g.8. Find the inverse Laplace transform of $\frac{1}{s^3(s^2+a^2)}$

Soln.

Let $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin au\,du$ $\quad \left(\because L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at\right)$

$= \left|-\frac{1}{a^2}\cos au\right|_0^t = \frac{1}{a^2}(1-\cos at)$

$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \frac{1}{a^2}(1-\cos au)\,du = \frac{1}{a^2}\left[|u|_0^t - \left|\frac{\sin au}{a}\right|_0^t\right]$

$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$

$L^{-1}\left\{\frac{1}{s^3(s^2+a^2)}\right\} = \int_0^t \frac{1}{a^2}\left[u - \frac{\sin au}{a}\right]du$

$= \frac{1}{a^2}\left[\left|\frac{u^2}{2}\right|_0^t + \left|\frac{\cos au}{a^2}\right|_0^t\right]$

$= \frac{1}{a^2}\left(\frac{t^2}{2} + (\cos at - 1)\frac{1}{a^2}\right)$ Ans

H.A Find the inverse Laplace transform of $\log\frac{s+1}{s-1}$