Gauss Divergence Theorem

(Relation between Surface and Volume integrals)

Statement: If $\vec{F}$ is a continuously differentiable vector function in the region E bounded by the closed Surface S, then

$$\int_S \vec{F}\cdot\hat{N}\,ds = \int_E \operatorname{div}\vec{F}\,dv$$

i.e

$$\iint_S \vec{F}\cdot\hat{N}\,ds = \iiint_E \operatorname{div}\vec{F}\,dv \quad \rightarrow (1)$$

Where $\hat{N}$ is the unit external normal vector at any point of S

$$\hat{N} = \cos\alpha\,\hat{I} + \cos\beta\,\hat{J} + \cos\gamma\,\hat{k}$$

and $\vec{F}(\vec{R}) = f_1(x,y,z)\hat{I} + f_2(x,y,z)\hat{J} + f_3(x,y,z)\hat{k}$

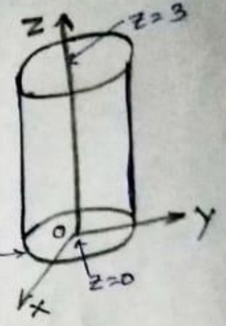
$\therefore$ eqn (1) Can be written as

$$\iint_S (f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy) = \iiint_E \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz \quad \rightarrow (2)$$

$\therefore$ Projection of δs on xy plane be $\delta s \cos\gamma = \delta x\,\delta y$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\hat{I}\frac{\partial}{\partial x} + \hat{J}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\hat{I} + f_2\hat{J} + f_3\hat{k}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f}{\partial z}$$

e.g 1. Using Gauss Divergence theorem, Evaluate $\int_S \vec{F}\cdot d\vec{s}$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$



Soln: Gauss Divergence Theorem,

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_V \operatorname{div}\vec{F}\, dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dx\,dy\,dz$$

$$= \int_{x=-2}^{2} \left\{ \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\int_{z=0}^{3} (4-4y+2z)\,dz \right] dy \right\} dx$$

$$= \int_{-2}^{2} \left\{ \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left| 4z - 4yz + z^2 \right|_0^3 dy \right\} dx$$

$$= \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\,dy \right] dx$$

$$= \int_{-2}^{2} \left[21y - 6y^2 \right]_{-\sqrt{4-x^2}}^{\sqrt{4+x^2}} dx$$

$$= 42\int_{-2}^{2} \sqrt{4-x^2}\, dx = 84 \int_0^2 \sqrt{4-x^2}\, dx$$

$$= 84 \left| \frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_0^2 = 84\pi$$

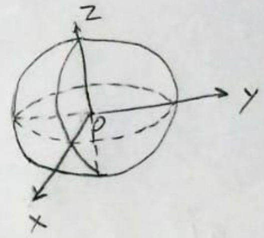
e.g.2. Using Divergence theorem, Evaluate

$\int_S \vec{A} \cdot d\vec{s}$, where $\vec{A} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$

we know $\iint_S \vec{A} \cdot d\vec{s} = \iiint_V \operatorname{div} \vec{A} \, dv$

$\operatorname{div} A = \nabla \cdot A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}\right)$

$\iint_S \vec{A} \cdot d\vec{s} = \iiint_V (3x^2+3y^2+3z^2) \, dx\,dy\,dz$

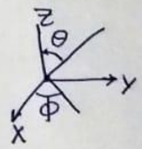


Changing to Spherical polar Co-ordinate $(x, y, z) \longrightarrow (r, \theta, \phi)$

putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} 3r^2 \left\{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right\} r^2 \sin\theta \, dr\,d\theta\,d\phi$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin\theta \, dr\,d\theta\,d\phi$$



$$= 3\frac{a^5}{5} \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= 3\frac{a^5}{5} \int_0^{2\pi} \left|-\cos\theta\right|_0^{\pi} d\phi = \frac{6a^5}{5} \int_0^{2\pi} d\phi$$

$$= \frac{12\pi a^5}{5}$$