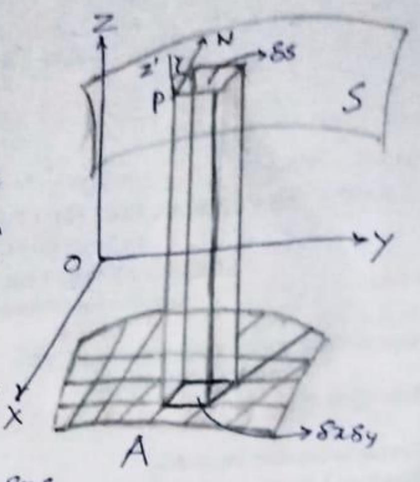
Area of the curved surface

Let us consider a surface

$S: \; z = f(x,y) \quad \ldots(i)$



P = any point on it

A = projection of S in xy plane

let us divide A into elementary area by drawing lines parallel to X & Y axes.

$\delta x \delta y$ = elementary area

on $\delta x \delta y$ erect a cylinder whose generator is parallel to Z-axis

δs = elementary surface on S

projection of δs on xy plane be $\delta x \delta y$

i.e. $\delta x \delta y = \delta s \cos\gamma \quad \ldots(ii)$

$$\delta s = \frac{\delta x \delta y}{\cos\gamma}$$

where γ is the angle between the xy plane and tangent plane to S at P i.e. Z-axis and the normal to S at P $\angle Z'PN = \gamma$

$\because \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the d.rs of the normal to the surface $f(x,y,z) = 0$

$\therefore$ drs of the normal to the surface $z = f(x,y)$ be $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1$

i.e. $F = (f(x,y) - z)$ or $(z - f(x,y))$

$-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1,$

drs. of z axis $= 0, 0, 1.$

$$\therefore \cos\gamma = \frac{1}{\sqrt{\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}}}$$

$\therefore$ from (ii)
$$\delta s = \frac{\delta x \delta y}{\cos\gamma} = \frac{\delta x \delta y}{\frac{1}{\left\{(\partial z/\partial x)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{1/2}}} = \frac{\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}} \delta x \delta y}{1}$$

$$S = \lim_{\delta s \to 0} \sum \delta s = \iint_A \sqrt{\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}} \, dx\,dy$$

$\therefore$ The Surface area where projection is taken in xy-plane

$$\boxed{S = \iint_A \sqrt{\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}}\, dx\,dy}$$

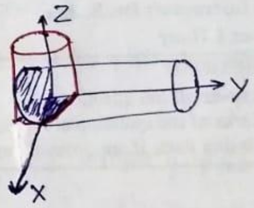
Illy

$$\boxed{S = \iint_B \sqrt{\left\{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right\}}\, dy\,dz}$$ Projection taken in yz plane

Illy

$$\boxed{S = \iint_C \sqrt{\left\{\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right\}}\, dz\,dx}$$ Projection taken in zx plane

e.g 1. Find the surface of $x^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=a^2$



Soln $S = \iint_A \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} dx\,dy$ ..(i)

we have $x^2+z^2=a^2 \rightarrow (*)$

$y=0$ to $\sqrt{a^2-x^2}$
$x=0, a$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0$$

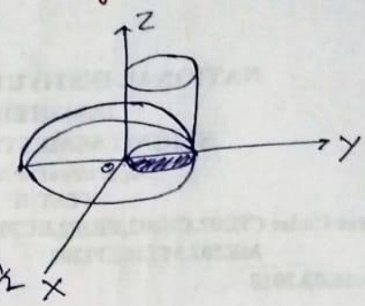
$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} = \left(\frac{x^2}{z^2}+1\right)^{1/2} = \left(\frac{x^2+z^2}{z^2}\right)^{1/2} = \left(\frac{a^2}{z^2}\right)^{1/2} = \frac{a}{z}$$

projection is a quadrant circle, $S = 8$ [Surface area of the upper portion of $(*)$ lying within the cylinder in the +ve quadrant]

Reqd Surface: $S = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2}}\, dx\,dy = 8\int_0^a \frac{a}{\sqrt{a^2-x^2}} \left|y\right|_0^{\sqrt{a^2-x^2}} dx$

$$= 8a \int_0^a dx = 8a^2 \text{ sq. unit}$$

eg 2. Find the area of the surface of the sphere $x^2+y^2+z^2=a^2$ which lies inside the cylinder $x^2+y^2=ay$



$$S = \iint_A \left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]^{1/2} dx\,dy$$

$$x^2+y^2+z^2=a^2$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -y/z$$

$$\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]^{1/2} = \left[\frac{x^2+y^2+z^2}{z^2}\right]^{1/2}$$

$$= \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

Reqd surface

i.e $S = 4\iint_A \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$ over half of the circle $x^2+y^2=ay$.

Changing into polar co-ordinates $x = r\cos\theta,\ y=r\sin\theta$

$x^2+y^2=ay \Rightarrow r^2 = ar\sin\theta$

$r = a\sin\theta$

$$S = 4a\int_0^{\pi/2}\int_{r=0}^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$$= -4a\int_0^{\pi/2} d\theta \left|\sqrt{a^2-r^2}\right|_0^{a\sin\theta}$$

$$\left\{\begin{array}{l} a^2-r^2=t^2 \\ -2r\,dr = 2t\,dt \\ \frac{r\,dr}{\sqrt{a^2-r^2}} = \frac{t}{t}\,dt = dt \end{array}\right.$$

$$= -4a\int_0^{\pi/2} (a\cos\theta - a)\,d\theta$$

$$= -4a^2 \left\{\left|\sin\theta\right|_0^{\pi/2} - \left|\theta\right|_0^{\pi/2}\right\}$$

$$= -4a^2\left[1-\pi/2\right] = (2-\pi)(-2a^2)$$

$$= 2a^2(\pi-2) \quad \text{sq unit}$$

eg 3. find the area of the surface $x^2+y^2+z^2=a^2$ inside the surfaces $(x^2+y^2)^2 = a^2(x^2-y^2)$

Sol^n $S = \iint_A \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} dx\,dy.$

here $x^2+y^2+z^2=a^2 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z};\ \frac{\partial z}{\partial y} = -y/z$

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} = \left[\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1\right]^{1/2} = \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

Symmetrical about all octants

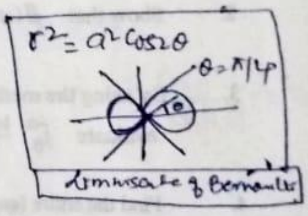
Reqd surface $S = 8 \iint_A \frac{a}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$

Changing to polar co-ordinate $x = r\cos\theta,\ y = r\sin\theta$

$(x^2+y^2)^2 = a^2(x^2-y^2)$

$r^4 = a^2 r^2 \cos 2\theta$

$r = a\sqrt{\cos 2\theta}$



$$S = 8 \int_{\theta=0}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{a\, r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$$= 8a \int_0^{\pi/4} d\theta\, (-) \left|\sqrt{a^2-r^2}\right|_0^{a\sqrt{\cos 2\theta}}$$

$$= -8a \int_0^{\pi/4} \left(\sqrt{a^2 - a^2\cos 2\theta} - a\right) d\theta$$

$$= -8a^2 \int_0^{\pi/4} \left(\sqrt{1-\cos 2\theta} - 1\right) d\theta$$

$$= -8a^2 \int_0^{\pi/4} \left(\sqrt{2}\sin\theta - 1\right) d\theta$$

$$= 8a^2 \int_0^{\pi/4} \left(1 - \sqrt{2}\sin\theta\right) d\theta$$

$$= 8a^2 \left(\pi/4 + \sqrt{2}\left|\cos\theta\right|_0^{\pi/4}\right) = 8a^2\left(\frac{\pi}{4} + \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1\right)\right)$$

$$= 8a^2 \left(\frac{\pi}{4} + \left(1 - \frac{1}{\sqrt{2}}\right)\right) \text{ Sq unit}$$

e.g. Calculate the volume of the solid bounded by the surface $x=0$, $y=0$, $x+y+z=1$ and $z=0$

Reqd volume $= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dx\,dy\,dz$

$= \int_0^1 \int_0^{1-x} (1-x-y)\,dx\,dy = \int_0^1 \left| y - xy - \frac{y^2}{2} \right|_0^{1-x} dx$

$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$

$= \int_0^1 \left[1-x-x+x^2-\frac{1}{2}-\frac{x^2}{2}+\frac{2x}{2} \right] dx$

$= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx \quad = \left| \frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6} \right|_0^1$

$\frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$ cubic unit

e.g. Evaluate the integrale by changing to spherical polar co-ordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} = I$$

Soln For changing to spherical polar co-ordinate putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{\sqrt{(1-r^2)}} r^2 \sin\theta \, dr\,d\theta\,d\phi \qquad \because J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin\theta \, dr\,d\theta\,d\phi$

$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right) dr \times \sin\theta\,d\theta\,d\phi$

$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\,d\theta \left[\sin^{-1}r - \frac{1}{2} r\sqrt{1-r^2} - \frac{1}{2}\sin^{-1}r \right]_0^1$

$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\,d\theta \,(\pi/2 - \pi/4) = \frac{\pi}{4} \int_0^{\pi/2} d\phi \,(-\cos\theta)_0^{\pi/2} d\phi$

$= \frac{\pi}{4} \int_0^{\pi/2} d\phi = \frac{\pi^2}{8}$

e.g. Find the volume comm to the cylinder $x^2+y^2=a^2$, $x^2+z^2=a^2$

Soln Reqd volume $= \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dy\,dz = \frac{16a^3}{3}$ cubic unit