Triple Integral

The symbol $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz = \int_{x_1=a}^{x_2=b}\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)}\int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x,y,z)\,dx\,dy\,dz.$

$$= \int_{x_1=a}^{x_2=b}\left[\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)}\left\{\int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x,y,z)\,dz\right\}dy\right]dx$$

eg1. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz.$

Solution The given Integral can be written as

$$\int_{z=-1}^{1}\left[\int_{x=0}^{z}\left\{\int_{y=x-z}^{x+z}(x+y+z)\,dy\right\}dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left\{\left|xy+\frac{y^2}{2}+yz\right|_{x-z}^{x+z}\right\}dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left\{x(x+z-x+z)+\tfrac{1}{2}\left[(x+z)^2-(x-z)^2\right]+2z^2\right\}dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}(2zx+2zx+2z^2)\,dx\right]dz$$

$$= \int_{-1}^{1}\left[\left|4z\frac{x^2}{2}+2z^2x\right|_{0}^{z}dz = \int_{-1}^{1}(2z^3+2z^3)\,dz = \int_{-1}^{1}4z^3\,dz\right.$$

$$= \left|4\frac{z^4}{4}\right|_{-1}^{1} = 0$$

Eg2. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r\,dr\,d\theta\,dz$

Sol^n $\int_{\theta=0}^{\pi/2}\left\{\int_{r=0}^{a\sin\theta}\left[\int_0^{(a^2-r^2)/a} r\,dz\right]dr\right\}d\theta$

$= \int_0^{\pi/2}\left\{\int_0^{a\sin\theta} r\frac{(a^2-r^2)}{a}\,dr\right\}d\theta = \int_0^{\pi/2}\left[\int_0^{a\sin\theta}\left(ar-\frac{r^3}{a}\right)dr\right]d\theta$

$= \int_0^{\pi/2}\left[\left|\frac{ar^2}{2}\right|_0^{a\sin\theta} - \left|\frac{r^4}{4a}\right|_0^{a\sin\theta}\right]d\theta$

$= \int_0^{\pi/2}\left[\frac{a^3\sin^2\theta}{2} - \frac{a^3\sin^4\theta}{4}\right]d\theta$

$= \frac{a^3}{2}\int_0^{\pi/2}\sin^2\theta\,d\theta - \frac{a^3}{4}\int_0^{\pi/2}\sin^4\theta\,d\theta$

$= \frac{a^3}{2}\frac{\Gamma 3/2\,\Gamma 1/2}{2\Gamma 2} - \frac{a^3}{4}\frac{\Gamma 5/2\,\Gamma 1/2}{2\Gamma 3}$

$= \frac{a^3}{4}\frac{1}{2}\pi - \frac{a^3}{16}\,\frac{3}{2}\frac{1}{2}\pi$

$= \frac{a^3}{8}\pi\left(1-\frac{3}{8}\right) = \frac{5\pi a^3}{64}$ Ans.

eg8. Evaluate the Integral

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$$

Soln Given Integral can be written as

$$\int_{x=0}^{\log 2} e^{x}dx \times \int_{y=0}^{x} e^{y}dy \times \int_{z=0}^{x+\log y} e^{z}dz$$

$$= \int_0^{\log 2} e^{x}dx \times \int_0^{x} e^{y}dy\ [e^{x}.e^{\log y}-1] = \int_0^{\log 2} e^{x}dx \times \int_0^{x} e^{y}(ye^{x}-1)dy$$

$$= \int_0^{\log 2} e^{x}dx \times \int_0^{x}(ye^{y}e^{x}-e^{y})dy$$

$$= \int_0^{\log 2}\left[e^{2x}\left\{\int_0^{x} ye^{y}dy\right\}-e^{x}\int_0^{x}e^{y}dy\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}\left|ye^{y}-e^{y}\right|_0^{x}-e^{x}\left|e^{y}\right|_0^{x}\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}(xe^{x}-e^{x}+1)-e^{x}(e^{x}-1)\right]dx$$

$$= \int_0^{\log 2}\left[xe^{3x}-e^{3x}+e^{2x}-e^{2x}+e^{x}\right]dx$$

$$= \left|\frac{xe^{3x}}{3}-\frac{e^{3x}}{9}\right|_0^{\log 2}-\left|\frac{e^{3x}}{3}\right|_0^{\log 2}+\left|e^{x}\right|_0^{\log 2}$$

$$= \frac{8}{3}\log 2-\frac{8}{9}+\frac{1}{9}-\frac{8}{3}+\frac{1}{3}+2-1$$

$$= \frac{8}{3}\log 2-\frac{19}{9}\ \text{Ans}$$

Q.4. Show that $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8}$.

Sol. The given Integral can be written as

$$\int_{x=0}^1 \left[\int_{y=0}^{\sqrt{(1-x^2)}} \left\{ \int_{z=0}^{\sqrt{(1-x^2-y^2)}} \frac{dz}{\sqrt{[(1-x^2-y^2)-z^2]}} \right\} dy \right] dx$$

$$= \int_0^1 \int_0^{\sqrt{(1-x^2)}} \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dz}{\sqrt{[(\sqrt{(1-x^2-y^2)})^2-z^2]}} \right\} dy \Big] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{(1-x^2)}} \left\{ \left| \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right|_0^{\sqrt{(1-x^2-y^2)}} \right\} dy \right] dx$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}}$$

$$= \int_0^1 \left[\int_0^{\sqrt{(1-x^2)}} \left(\sin^{-1} 1 - \sin^{-1} 0 \right) dy \right] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{(1-x^2)}} \pi/2 \; dy \right] dx = \int_0^1 \pi/2 \sqrt{1-x^2}\; dx$$

Putting $x = \cos\theta$

$$= \pi/2 \, (-) \int_{\pi/2}^{0} \sin^2\theta \, d\theta = \frac{\pi}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \pi/2 \, \frac{\Gamma 3/2 \, \Gamma 1/2}{2\,\Gamma 2} = \frac{\pi}{2} \frac{1}{2} \frac{\pi}{2} = \frac{\pi^2}{8} \text{ Ans.}$$