Change the order of Integration

Let $$\int_{x_1=a}^{x_2=b}\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\,dx \rightleftarrows \int_{y_1=c}^{y_2=d}\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx\,dy$$

Working rule

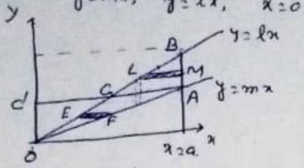
$$\int_a^b\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx.$$

1. Draw the figure from the given limits.
2. Find the point of intersection (if any)
3. Mark the region (or Area) integration
4. Draw straight line parallel to x-axis from the point of intersection and divide the region in different parts
5. Draw elementary stripe parallel to the straight line (or x-axis) drawn in each part.
6. Find where the extremities of the stripe lies and that will be the limit of x in term of y
7. Find the variation of y in each part, that will be the limit of y in term of constant
8. If there will be different parts add such integrals of each part

Illy $$\int_c^d\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx\,dy$$

eg.1. Change the order of Integration $\int_{x=0}^{a}\int_{y=mx}^{lx} v\,dx\,dy$.

$y=mx,\ y=lx,\ x=0,\ x=a$



Point of intersection $A(a, ma)$ & $B(a, la)$

Area of integration OABO

Let us draw straight line AC ‖ to x-axis which divides the region of integration into two parts, they are OAC & CAB

Let us draw elementary stripe EF & LM parallel to AC in OAC & CAB respectively.

In OAC

E lies on $x = y/l$

F lies on $x = y/m$

EF varies from (O to A)

$y = 0$ to ma

In CAB

L lies on $x = y/l$

M lies on $x = a$

y varies from (A to B)

i.e. ma to la.

$$\int_0^a \int_{mx}^{lx} v\,dx\,dy = \int_{y=0}^{ma}\int_{x=y/l}^{y/m} v\,dx\,dy + \int_{y=ma}^{la}\int_{x=y/l}^{a} v\,dx\,dy$$

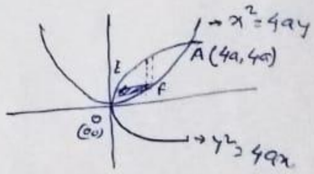
eg2 Change the order of integration in

$$\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx, \text{ hence evaluate it}$$

Given limits $y = x^2/4a \Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$

$x = 0,\ x = 4a.$



$y^2 = 4ax \Rightarrow \frac{x^4}{16a^2} = 4ax$

$\Rightarrow x^3 = 64a^3$

$\Rightarrow x = 4a$

& $y = 4a.$

Pt of intersection $A(4a, 4a)$

Area of Integration OAO

EF elementary stripe

$x = \frac{y^2}{4a},\ x = 2\sqrt{ay}$

y varies from 0 to $4a$.

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy.$$

$$= \int_0^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy = \int_0^{4a}\left(2a^{1/2}y^{1/2} - \frac{y^2}{4a}\right)dy$$

$$= \left(2a^{1/2}\,y^{3/2}\times\tfrac{2}{3} - \frac{y^3}{12a}\right)_0^{4a} = 2a^{1/2}\times\frac{2}{3}\,4^{3/2}a^{3/2} - \frac{4^3a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ Sq unit.}$$

eg 3.
Evaluate the double Integral

$$\int_0^{2a}\int_{y^2/4a}^{3a-y} dx\,dy$$ after changing the order of integration

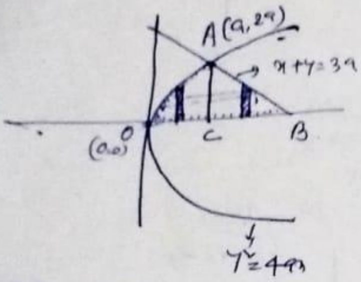
$$\int_{y=0}^{2a}\int_{x=y^2/4a}^{3a-y} dx\,dy$$

given limits

$x = \frac{y^2}{4a} \Rightarrow y^2 = 4ax.$

$x = 3a - y \Rightarrow x + y = 3a.$

$y = 0,\ y = 2a.$



Area of Integration
OABO

AC ∥ y. axis

In the region OACO

$y = 0,\ y = 2\sqrt{ax}$

$x = 0$ to a.

In the region ACBA

$y = 0,\ y = 3a - x$

$x = a$ to $3a$.

$$\int_0^{2a}\int_{x=y^2/4a}^{3a-y} dx\,dy = \int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} dy\,dx + \int_{x=a}^{3a}\int_{y=0}^{3a-x} dy\,dx$$

$$= \int_0^a |y|_0^{2\sqrt{ax}}\,dx + \int_a^{3a} |y|_0^{3a-x}\,dx$$

$$= \int_0^a 2\sqrt{ax}\,dx + \int_a^{3a}(3a-x)\,dx$$

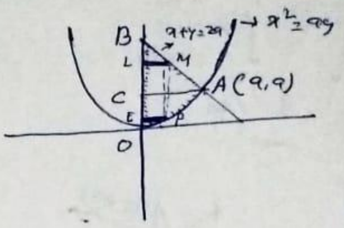
$$= \left|2a^{1/2}x^{3/2}\times \tfrac{2}{3}\right|_0^a + \left|3ax - \frac{x^2}{2}\right|_a^{3a}$$

$$= \frac{4}{3}a^{1/2}a^{3/2} + \left[3a\times 3a - \frac{9a^2}{2} - 3a^2 + \frac{a^2}{2}\right]$$

$$= \frac{4}{3}a^2 + \left[6a^2 - \frac{8a^2}{2}\right] = \frac{4}{3}a^2 + 2a^2 = \frac{10a^2}{3}\ \text{Ans}$$

eg4 Evaluate the following Integrals by changing the order of integration

$$\int_{x=0}^{a}\int_{y=x^2/a}^{2a-x} xy\,dx\,dy.$$



Limit

$y = \frac{x^2}{a} \Rightarrow x^2 = ay,$

$y = 2a - x \Rightarrow x + y = 2a.$

$x = 0,\ x = a.$

Area of Integration OABO

In the region OAC

$x = 0,\quad x = \sqrt{ay}$

$y = 0$ to a.

In the region CAB

$x = 0,\quad x = 2a - y$

$y = a$ to $2a$

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy = \int_0^a \int_{x=0}^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a}\int_{x=0}^{2a-y} xy\,dx\,dy$$

$$= \int_0^a \left| y\frac{x^2}{2} \right|_0^{\sqrt{ay}} dy + \int_a^{2a} \left| \frac{x^2}{2}y \right|_0^{2a-y} dy$$

$$= \int_0^a \frac{y\,ay}{2}\,dy + \int_0^{2a} \frac{(2a-y)^2}{2}y\,dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_0^{2a} [4a^2y - 4ay^2 + y^3)\,dy$$

$$= \frac{a}{2}\left|\frac{y^3}{3}\right|_0^a + \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_0^{2a}$$

$$= \frac{a}{2}\frac{a^3}{3} + \frac{1}{2}\left[\frac{16a^4}{2} - \frac{32a^4}{3} + \frac{16a^4}{4}\right] = \frac{a^4}{6} + \frac{1}{2}a^4\frac{5}{12}$$

$$= \frac{3a^4}{8} \text{ Ans}$$

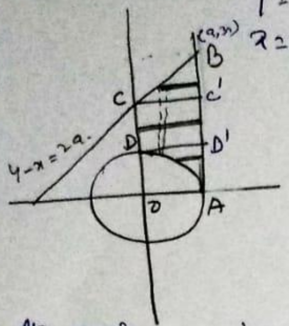
Q5 Change the order of integration in

$$\int_{x=0}^{a}\int_{y=\sqrt{(a^2-x^2)}}^{x+2a} f(x,y)\,dx\,dy.$$

Given limits $y=\sqrt{(a^2-x^2)} \Rightarrow x^2+y^2=a^2$

$y = x+2a \Rightarrow y-x=2a.$

$x=0$, $x=a.$



Area of Integration ABCDA

Let us draw straight line parallel to x-axis DD' & CC' in the region

Three different region are DAD'D, DD'C'C, CC'BC

In the region DAD'D

$x=\sqrt{a^2-y^2}$, $x=a$

$y=0$ to a.

In the region DD'C'C

$x=0$, $x=a$

$y=a$ to $2a$

In the region CC'BC

$x=y-2a$; $x=a$

$y=2a$ to $3a$.

$$\therefore \int_{x=0}^{a}\int_{y=\sqrt{(a^2-x^2)}}^{x+2a} f(x,y)\,dx\,dy = \int_{y=0}^{a}\int_{x=\sqrt{(a^2-y^2)}}^{a} f(x,y)\,dx\,dy + \int_{y=a}^{2a}\int_{x=0}^{a} f(x,y)\,dx\,dy.$$

$$+ \int_{y=2a}^{3a}\int_{x=y-2a}^{a} f(x,y)\,dx\,dy$$