Double Integration

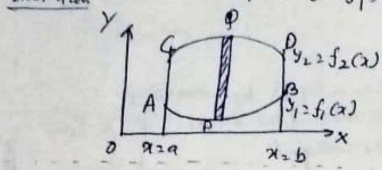
The symbol $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$ is said to be the Double Integration

Double integration can be evaluated as follows:

Case- I When $\int_{x_1=a}^{x_2=b}\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\,dx.$

then $\int_{x_1=a}^{x_2=b}\left\{\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\right\}dx.$

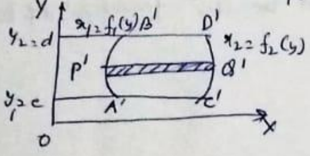
Illustration



Inner Integral means Integration along the Stripe PQ from P to Q
outer Integral means such PQ slided from AC to BD covering the area ABDC

Case II When $\int_{y_1=c}^{y_2=d}\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx\,dy.$

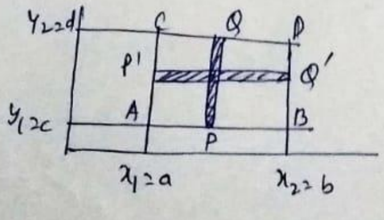
then $\int_{y_1=c}^{y_2=d}\left\{\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx\right\}dy.$



Inner Integral means Integration along the Stripe P'Q' from P' to Q'
outer Integral means such P'Q' slided from A'C' to B'D' covering the area A'C'D'B'

Case III When $\int_{x_1=a}^{x_2=b}\int_{y_1=c}^{y_2=d} f(x,y)\,dy\,dx$ then

$$\int_{x_1=a}^{x_2=b}\int_{y_1=c}^{y_2=d} f(x,y)\,dy\,dx = \begin{cases}\int_{x_1=a}^{x_2=b}\left\{\int_{y_1=c}^{y_2=d} f(x,y)\,dy\right\}dx\\[2ex] \int_{y_1=c}^{y_2=d}\left\{\int_{x_1=a}^{x_2=b} f(x,y)\,dx\right\}dy\end{cases}$$

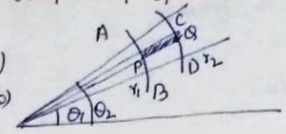


In this case the Integration gives the area of rectangle ABDC.

Integration In Poler. Co-ordinate

$$\int_{\theta=\theta_1}^{\theta_2} \left\{ \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r,\theta)\, dr \right\} d\theta$$

$AB: r_1 = f_1(\theta)$
$CD: r_2 = f_2(\theta)$

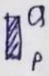


Inner Integral means the Integration is along PQ from P to Q
Outer integral means such PQ rotated from BD to AC covering the area ABDC

When limits are not given

1. Draw the region
2. Find the limit of y in terms of x for inner integral by drawing Stripe parallel to y axis from the extremites of the Stripe
3. Find the limit of x as constant for outer integral

$$\int_{x=a}^{b} \int_{y=f_1(x)}^{y_2=f_2(x)}$$



or 2. Find the limit of x in terms of y for inner integral by drawing Stripe parallel to x-axis from the extremites of the Strips
or 3 Find the limit of y as Constant for outer integral

$$\int_{y=c}^{d} \int_{x_1=f_1(y)}^{x_2=f_2(y)}$$

Note The limit of the Inner integral must be the function of the other (outer) e.g.1. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$$\int_0^a \left\{ \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx \right\} dy.$$

Let $a^2-y^2=\alpha^2$

$$= \int_0^a \left\{ \int_0^{\alpha} \sqrt{\alpha^2-x^2}\, dx \right\} dy$$

$x = \alpha \sin\theta$

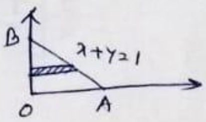
$$= \int_0^a \left\{ \int_0^{\pi/2} \alpha\cos\theta \times \alpha\cos\theta\, d\theta \right\} dy = \int_0^a \left\{ \int_0^{\pi/2} \alpha^2 \cos^2\theta\, d\theta \right\} dy$$

$$= \int_0^a \left(\alpha^2 \frac{\pi}{4}\right) dy = \frac{\pi}{4} \int_0^a (a^2-y^2)\, dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{2}\,\frac{a^3}{3} = \frac{\pi a^3}{6} \text{ Ans}$$

e.g.2. Evaluate $\iint_A (x^2+y^2)\, dx\,dy$ over the region A in the positive quadrant for which $x+y \le 1$

$$\int_{y=0}^{1} \left[\int_{x=0}^{1-y} (x^2+y^2)\, dx \right] dy$$

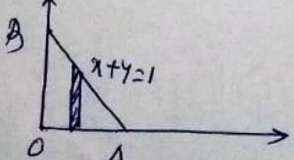


$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y} dy = \int_0^1 \left[\frac{(1-y)^3}{3} + (1-y)y^2\right] dy$$

$$= \int_0^1 \left\{\frac{(1-y)^3}{3} + y^2 - y^3\right\} dy = -\left[\frac{(1-y)^4}{3\times4}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1$$

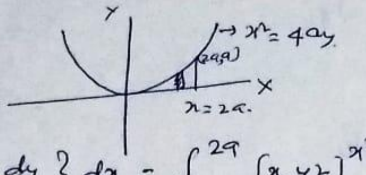
$$= \frac{1}{12} + \frac{1}{3} - \frac{1}{4} = \frac{1}{6} \text{ Ans}$$

Alternatively



$$\int_{x=0}^{1} \left\{ \int_{y=0}^{1-x} (x^2+y^2)\, dy \right\} dx.$$

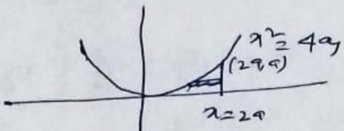
e.g.3 Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.



$$\int_{x=0}^{2a}\left\{\int_{y=0}^{x^2/4a} xy\,dy\right\}dx = \int_0^{2a}\left[x\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_0^{2a}\frac{x^5}{32a^2}\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2}\times\frac{2^6a^6}{6} = \frac{1}{3}a^4 \text{ Ans}$$

alternatively



$$\int_{y=0}^{a}\left\{\int_{x=2\sqrt{ay}}^{2a} xy\,dx\right\}dy = \int_0^a\left[\frac{x^2y}{2}\right]_{2\sqrt{ay}}^{2a} dy = \frac{1}{2}\int_0^a\left(4a^2y - 4ay^2\right)dy$$

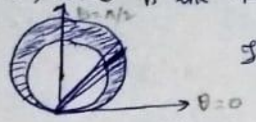
$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3}\right]_0^a \quad \left[\frac{{}^2\cancel{4}a^4}{\cancel{2}} - \frac{4a^4}{3}\right]\times\frac{1}{2}$$

$$= \frac{6a^4 - 4a^4}{3\times 2} = \frac{a^4}{3} \text{ Ans}$$

eg-4 Calculate $\iint r^3\,dr\,d\theta$, over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$.

Putting $\theta = \pi - \theta$, the eqns remain unchanged, ∴ both circles are symmetrical about $\theta = \pi/2$.

Putting $r = 0$, $\theta = 0$ is the tangent for both the circles.



The shaded portion is the reqd area.

$$\int_{\theta=0}^{\pi}\left\{\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\,dr\right\} d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{4}\int_0^{\pi}\left(4^4\sin^4\theta - 2^4\sin^4\theta\right)d\theta$$

$$= \frac{1}{4}\times 2^4\times 15\int_0^{\pi}\sin^4\theta\,d\theta = 60\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta = 120\times\frac{3\pi}{16}$$

$= 22.5\pi$ Ans.

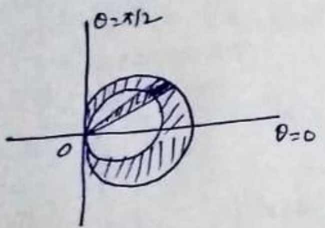
eg5 Evaluate $\iint r^3\,dr\,d\theta$, over the area between the circles $r = 2a\cos\theta$ and $r = 2b\cos\theta$ $(b<a)$

Soln Putting $\theta = -\theta$, eqn. of the curve remain unchanged then the curve is symmetrical about the initial line.

Here putting $\theta = -\theta$, the eqns remain unchanged

∴ both the circles are symmetrical about the initial line.

Putting $r = 0$, $\theta = \pi/2$ is the tangent.



$$\int_{\theta=-\pi/2}^{\pi/2}\int_{2b\cos\theta}^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left|\frac{r^4}{4}\right|_{2b\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\left(2^4a^4\cos^4\theta - 2^4b^4\cos^4\theta\right)d\theta = \frac{1}{4}2^4\int_{-\pi/2}^{\pi/2}(a^4-b^4)\cos^4\theta\,d\theta$$

$$= (a^4-b^4)\times\frac{1}{4}2^4\times 2\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{1}{4}\times 2^4\times 2\,(a^4-b^4)\frac{\Gamma 5/2\,\Gamma 1/2}{2\Gamma 3}$$

$$= \frac{1}{4}\times 2^4\times 2\,(a^4-b^4)\frac{3\pi}{16} = \frac{3}{2}\pi\,(a^4-b^4)$$ Ans.