Differentiation under the sign of Integration
(Differentiation under the Integral Sign)

Let $F(\alpha) = \int_a^b f(x,\alpha)\,dx$ ①

Where a, b are constants or the function of α

To find $\frac{dF(\alpha)}{d\alpha}$ when it exists

i.e. $\frac{dF}{d\alpha}(\alpha) = \frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx$ —②

When it is not always possible to first integrate and then find the derivative, we use Leibnitz's rule to solve such problems.

(I) If $f(x,\alpha)$ and $\frac{\partial f(x,\alpha)}{\partial\alpha}$ be continuous functions of x and α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx$$

where a, b are constants independent of α

(II) Leibnitz's rule for variable limit of integration

If $f(x,\alpha)$ & $\frac{\partial f(x,\alpha)}{\partial\alpha}$ be continuous function of x and α then

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx - \frac{d\phi}{d\alpha} f[\phi(\alpha),\alpha] + \frac{d\psi}{d\alpha} f[\psi(\alpha),\alpha]$$

Provided $\phi(\alpha)$ and $\psi(\alpha)$ possesses continuous first order derivative w.r.t. α

eg1. Evaluate $\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$, $\alpha \geq 0$, using the method of differentiation under the sign of integration

$F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$ ① then $\frac{dF}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\log x}\right)dx = \int_0^1 \frac{x^\alpha \log x}{\log x}\,dx$

$= \int_0^1 x^\alpha\,dx = \left|\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1 = \frac{1}{1+\alpha}$.

$\therefore F'(\alpha) = \frac{1}{1+\alpha} \Rightarrow F(\alpha) = \log(1+\alpha) + c$ ②

from ① $\alpha = 0 \Rightarrow F(0) = 0$

from ② $\alpha = 0 \Rightarrow F(0) = c = 0 \Rightarrow c = 0$

$\therefore$ from ② $F(\alpha) = \log(1+\alpha)$

eg 2. If $y = \int_0^x f(t) \sin k(x-t)\,dt$, show that $\frac{d^2y}{dx^2} + k^2 y = k f(x)$

we have $y = \int_0^x f(t) \sin k(x-t)\,dt$

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x f(t) \sin k(x-t)\,dt =$$

$$= \int_0^x \frac{\partial}{\partial x} f(t) \sin k(x-t)\,dt - \frac{d}{dx}(0) + \frac{d}{dx}(x)\left[f(x) \sin k(x-x)\right]$$

$$\frac{dy}{dx} = \int_0^x k f(t) \cos k(x-t)\,dt$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\int_0^x k f(t) \cos k(x-t)\,dt$$

$$= k\left\{\int_0^x \frac{\partial}{\partial x} f(t) \cos k(x-t)\,dt - \frac{d}{dx}(0) + \frac{dx}{dx}\left[f(x) \cos k(x-x)\right]\right\}$$

$$= -k^2 \int_0^x f(t) \sin k(x-t)\,dt + k f(x) = -k^2 y + k f(x)$$

$\frac{d^2y}{dx^2} + k^2 y = k f(x)$, proved.

eg 3. By differentiating under the integral sign, find $\frac{d^2F}{d\alpha^2}$ if $F(\alpha) = \int_{1/\alpha}^{2/\alpha} \frac{\sin \alpha x}{x^2}\,dx$, $\alpha \neq 0$

$$F(\alpha) = \int_{1/\alpha}^{2/\alpha} \frac{\sin \alpha x}{x^2}\,dx \quad (1)$$

$$\frac{dF}{d\alpha} = \frac{d}{d\alpha}\left[\int_{1/\alpha}^{2/\alpha} \frac{\sin \alpha x}{x^2}\,dx\right] = \int_{1/\alpha}^{2/\alpha} \frac{\partial}{\partial\alpha}\frac{\sin \alpha x}{x^2}\,dx - \frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)\left[\frac{\sin \alpha \cdot \frac{1}{\alpha}}{1/\alpha^2}\right] + \frac{d}{d\alpha}\left(\frac{2}{\alpha}\right)\left(\frac{\sin \alpha \cdot \frac{2}{\alpha}}{4/\alpha^2}\right)$$

$$= \int_{1/\alpha}^{2/\alpha} \frac{x \cos \alpha x}{x^2}\,dx + \frac{1}{\alpha^2}\cdot\alpha^2 \sin 1 - \frac{2}{\alpha^2}\cdot\frac{\alpha^2}{4}\sin 2$$

$$= \int_{1/\alpha}^{2/\alpha} \frac{x \cos \alpha x}{x^2}\,dx + \sin 1 - \frac{1}{2}\sin 2 \quad (2)$$

$$\frac{d^2F}{d\alpha^2} = \frac{d}{d\alpha}\int_{1/\alpha}^{2/\alpha} \frac{x \cos \alpha x}{x^2}\,dx = \int_{1/\alpha}^{2/\alpha} \frac{\partial}{\partial\alpha}\frac{x \cos \alpha x}{x^2}\,dx - \frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)\left(\frac{\cos \alpha \frac{1}{\alpha}}{1/\alpha}\right) + \frac{d}{d\alpha}\left(\frac{2}{\alpha}\right)\left(\frac{\cos \alpha \cdot \frac{2}{\alpha}}{2/\alpha}\right)$$

$$= -\int_{1/\alpha}^{2/\alpha} \frac{x \sin \alpha x}{x}\,dx + \frac{1}{\alpha^2}\cdot\alpha \cos 1 - \frac{2}{\alpha^2}\cdot\frac{\alpha}{2}\cos 2$$

$$= \left|\frac{\cos \alpha x}{\alpha}\right|_{1/\alpha}^{2/\alpha} + \frac{1}{\alpha}\cos 1 - \frac{1}{\alpha}\cos 2$$

$$= \frac{\cos 2}{\alpha} - \frac{\cos 1}{\alpha} + \frac{1}{\alpha}\cos 1 - \frac{1}{\alpha}\cos 2 = 0 \text{ Ans}$$

eg4. **Evaluate**

$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx$ by differentiating under the integral sign

Given that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Sol: Let $f(a) = \int_0^\infty e^{-(x^2+a^2/x^2)}\,dx$. ①

$$\frac{df}{da} = \frac{d}{da}\int_0^\infty e^{-(x^2+a^2/x^2)}dx = \int_0^\infty \frac{\partial}{\partial a} e^{-(x^2+a^2/x^2)}dx = \int_0^\infty e^{-(x^2+a^2/x^2)} \times -\frac{2a}{x^2}\,dx$$

• putting $x = a/y$ i.e. $y = a/x$.

$$\int_\infty^0 e^{-(a^2/y^2+y^2)} \times -\frac{2y^2}{a}\left(-\frac{a}{y^2}\right)dy = \int_0^\infty e^{-(y^2+a^2/y^2)}(-2)\,dy = -2f(a)$$

$$\frac{df}{da} = -2f(a) \Rightarrow \frac{df}{f(a)} = -2\,da. \Rightarrow \log f(a) = -2a+\log c$$

$$\log\frac{f(a)}{c} = -2a \Rightarrow \frac{f(a)}{c} = e^{-2a} \Rightarrow f(a) = ce^{-2a}. \quad ②$$

from ① $a=0$ $f(0) = \sqrt{\pi}/2$

from ② $f(0) = c$

$\therefore$ $f(a) = \sqrt{\pi}/2\, e^{-2a}$ Ans.

eg.5. If $F(\alpha) = \int_0^{\alpha^2}\tan^{-1}\left(\frac{x}{\alpha}\right)dx$. using differentiating under integral sign find $\frac{dF}{d\alpha}$

Sol:
$$\frac{dF}{d\alpha} = \frac{d}{d\alpha}\int_0^{\alpha^2}\tan^{-1}(x/\alpha)\,dx$$

$$= \int_0^{\alpha^2}\frac{\partial}{\partial\alpha}\tan^{-1}(x/\alpha)\,dx - \frac{d}{d\alpha}(0) + \frac{d}{d\alpha}(\alpha^2)\tan^{-1}\left(\frac{\alpha^2}{\alpha}\right)$$

$$= \int_0^{\alpha^2}\frac{1}{1+x^2/\alpha^2}\left(-x/\alpha^2\right)dx + 2\alpha\tan^{-1}\alpha$$

$$= \int_0^{\alpha^2} -\frac{x}{\alpha^2+x^2}\,dx + 2\alpha\tan^{-1}\alpha$$

$$= -\frac{1}{2}\left|\log(\alpha^2+x^2)\right|_0^{\alpha^2} + 2\alpha\tan^{-1}\alpha$$

$$= -\frac{1}{2}\left(\log(\alpha^2+\alpha^4) - \log\alpha^2\right) + 2\alpha\tan^{-1}\alpha$$

$$= 2\alpha\tan^{-1}\alpha - \frac{1}{2}\log\frac{(\alpha^2+\alpha^4)}{\alpha^2}$$

$$\frac{dF}{d\alpha} = 2\alpha\tan^{-1}\alpha - \frac{1}{2}\log(1+\alpha^2) \quad \text{Ans.}$$