Eigen Value and Eigen Vector

Let
$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots \quad \cdots \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}$$
be a Square matrix of order n

$A - \lambda I$ is called characteristic Matrix, where λ is a scalar and I is the nth order unit matrix

If the determinant of this matrix is equated to zero ie $|A - \lambda I| = 0$, is called characteristic equation and the root of this equation is called characteristic root or Laten root or Eigen Value of the matrix A.

A non zero vector X is called Eigen vector if
$$AX = \lambda X \Rightarrow AX = \lambda I X \Rightarrow (A - \lambda I)X = 0$$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called characteristic vector or Latenvector or Eigen vector corresponding to the Eigenvalue λ.

This can also be written as $X = [x_1, x_2, \cdots x_n]'$

Properties of Eigen values

1) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

2) If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}

3) If λ is an eigen value of an orthogonal matrix then $1/\lambda$ is also its eigen value.

[orthogonal matrix : if $AA' = A'A = I$, then square matrix A is orthogonal matrix]

<u>Properties of Eigen vectors</u>

1) The Eigen vector X of a matrix is not unique.

2) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ form a linearly independent set

3) If two or more eigen values are equal it may or may not be possible to get linearly independent eigen vectors, corresponding to the equal roots.

4) If $X_1' X_2 = 0$, then X_1 and X_2 are orthogonal vectors

<u>Theorem:</u> The Eigen vectors corresponding to distinct Eigen values of a matrix are linearly independent.

Eg Determine the characteristic roots and characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln $|A - \lambda I| = 0$ is the characteristic Eqn.

i.e. $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda)-16\right] + 6\left[-6(3-\lambda)+8\right] + 2\left[24-2(7-\lambda)\right] = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0 \Rightarrow \lambda = 0, 3, 15$

Eigen values or characteristic roots : 0, 3, 15.

Eigen vector: $(A - \lambda I)X = 0$

when $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 7x_2 - 4x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$

from the last two equations

$\frac{x_1}{21-16} = \frac{x_2}{-8+18} = \frac{x_3}{24-14}$

$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$ or $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$

$X_1 = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ Eigenvector

when $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$5x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 4x_2 - 4x_3 = 0$
$2x_1 - 4x_2 + 0x_3 = 0$

From last two eqs.

$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$ or $\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

$X_2 = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ Eigenvector

when $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$-7x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 - 8x_2 - 4x_3 = 0$
$+2x_1 - 4x_2 - 12x_3 = 0$

from last two eqn

$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$ or $\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

$X_3 = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ Eigen vector.

eg Find the Eigen values and Corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Soln: $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0 \rightarrow$ Characteristic eqn

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0 \Rightarrow \boxed{\lambda = 2, 2, 3}$ Eigen values.

For $\lambda = 2$

Eigen vector $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 10x_2 + 5x_3 = 0$

$-2x_1 - 5x_2 - 4x_3 = 0$

$3x_1 + 5x_2 + 5x_3 = 0$

From last two eqn

$$\frac{x_1}{\underset{-5}{-25+20}} = \frac{x_2}{\underset{-2}{-12+10}} = \frac{x_3}{\underset{5}{-10+15}}$$

$X_1 = \begin{bmatrix} -5k \\ -2k \\ 5k \end{bmatrix}$ or $\begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$ eigen vector

$X_2 = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$ eigen vector

For $\lambda = 3$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 + 10x_2 + 5x_3 = 0$

$-2x_1 - 6x_2 - 4x_3 = 0$

$3x_1 + 5x_2 + 4x_3 = 0$

From last two eqn

$$\frac{x_1}{\underset{=-4}{-24+20}} = \frac{x_2}{\underset{=-4}{-12+8}} = \frac{x_3}{\underset{=8}{-10+18}}$$

$X_3 = \begin{bmatrix} 1k \\ 1k \\ -2k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ Eigen vector.

X_1, X_2, X_3 are the eigen vectors

eg Find the Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solⁿ $A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}$ → Characteristic matrix

$|A - \lambda I| = 0 \quad \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$ → Characteristic Eqn

$$(-2-\lambda)\{-\lambda(1-\lambda) - 12\} - 2\{-2\lambda - 12\} - 3\{-4 + 1\cdot(1-\lambda)\} = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow \lambda = 5, -3, -3 \rightarrow \text{Eigen values}$$

Eigen vector : $(A - \lambda I)x = 0$

When $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

$$\frac{x_1}{20-12} = \frac{x_2}{6+10} = \frac{x_3}{-4-4}$$

$$\frac{x_1}{8} = \frac{x_2}{16} = \frac{x_3}{-8}$$

$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ Eigen vector

When $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$
$R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $x_1 + 2x_2 - 3x_3 = 0$

$X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ Eigenvector

Eigen vector

H.A. ① Find the eigen values and the corresponding eigenvectors of the matrix

(i) $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$, Ans: Eigenvalues $-1, -1, 3$
eigen vectors: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ Ans: Eigen values: $1, 2, 3$
Eigen vectors: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ Ans: Eigen values: $0, 1, 1$
Eigen vectors: $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

(iv) $D = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$ Ans: Eigen values: $-3, -6, 12$
Eigenvectors: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(v) $E = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Ans: Eigen values $1, 1, 4$
Eigen vectors: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$