System of linear equations:-

Let us consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= k_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= k_m. \end{aligned}\right\} \cdots (1)$$

The Coefficient Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix}$$

Augmented Matrix

$$K = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} & k_1 \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} & k_2 \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} & k_m \end{bmatrix}$$

Test the consistency

A Set of Simultaneous eqn which are Statisfied by at least one Set of values of the unknown

- Find the rank of A and K by reducing to triangular form by Elementary row transformation.

Let the rank of $A = r$ & rank of $K = r'$

i) if $r \neq r'$, the eqn are inconsistent and there is no solution.

ii) if $r = r' = n$, the eqn are Consistent and there is a unique Solution.

iii) if $r = r' < n$, the eqn are Consistent and there are infinite solution.

① Q Test the Consistency and solve
$x+y+z=6,\ x-y+2z=5,\ 3x+y+z=8,\ 2x-2y+3z=7$

The system of eqns can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -2 \\ 0 & -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -10 \\ -5 \end{bmatrix}$$

$$\begin{array}{l} R_3 - R_2 \\ R_4 - 2R_2 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \\ -3 \end{bmatrix}$$

$$R_4 - \tfrac{1}{3}R_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \\ 0 \end{bmatrix}$$

Rank of Coeff Matrix = 3 = Rank of Augmented Matrix
$\Rightarrow$ The system of eqns are Consistent.
$\Rightarrow$ eqn have a unique soln.

$x+y+z=6,\ -2y+z=-1,\ -3z=-9$
$\Rightarrow z=3,\ y=2,\ x=1$

(2) eg Test for consistency and solve

$x+y+z=6, \quad x+2y+3z=14, \quad x+4y+7z=30$

The system of eqns can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$R_2 - R_1$
$R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 24 \end{bmatrix}$$

$R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

Rank of coeffs matrix = 2 = Rank of Augmented matrix

$\Rightarrow$ The system of eqns are consistent and have infinite solns.

$x+y+z=6, \quad y+2z=8$

Let $z=c$ $\therefore$ $y=8-2c$ & $x=6-8+2c-c \Rightarrow x=c-2$

$z=c, x=c-2, y=8-2c$

For different values of c we get infinite soln.

(3) eg Apply rank test to test the consistency of the eqns

$2x+6y+11=0, \quad 6x+20y-6z+3=0, \quad 6y-18z+1=0$

The system of eqns can be written as:

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & +6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

$R_2 - 3R_1$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$$

$R_3 - 3R_2$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$$

Rank of Coeff matrix = 2

Rank of Augmented matrix = 3

$\therefore$ The system of eqns are inconsistent, hence no solution.

④

Ex Investigate for what values of λ, μ the simultaneous equation
$x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$.
have (i) no solution (ii) a unique solution (iii) infinite solution

The system of eqns can be written as

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$ where $A=\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}$, $K=\begin{bmatrix}1 & 1 & 1 & 6\\1 & 2 & 3 & 10\\1 & 2 & \lambda & \mu\end{bmatrix}$

$$\begin{array}{l}R_2-R_1\\R_3-R_2\end{array}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\0 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu-10\end{bmatrix}$$

(i) when $\lambda=3$ & $\mu\neq 0$

$\rho(A)=2$ & $\rho(K)=3$

$\because$ $\rho(A)\neq\rho(K)$, $\Rightarrow$ system of eqns for $\lambda=3$ & $\mu\neq 0$ is inconsistent hence no solution

(ii) $\lambda\neq 3$ & any value of μ.

$\rho(A)=3=\rho(K)$

$\Rightarrow$ System of eqns for $\lambda\neq 3$ & any value μ is consistent and have a unique solution.

(iii) $\lambda=3$ & $\mu=10$

$\rho(A)=2=\rho(K)$

$\Rightarrow$ System of eqns for $\lambda=3$ & $\mu=10$ is consisten and have infinite solution as the rank of $\rho(A)$ & $\rho(K)$ are equal but less than the number of unknown.

H.A

Test the consistency and solve.

$2x-y+3z=8$, $-x+2y+z=4$, $3x+y-4z=0$.

Ans $x=2=y=z$.

(5) For what values of μ the equation $x+y+z=1$, $x+2y+4z=\mu$, $x+4y+10z=\mu^2$ have a solution and solve them completely in each case

Let the system of equation can be written as

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \mu\\ \mu^2\end{bmatrix}$$

$$\underset{\sim}{\substack{R_2-R_1\\ R_3-R_1}}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 3 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \mu-1\\ \mu^2-1\end{bmatrix}$$

$$\underset{\sim}{R_3-3R_2}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \mu-1\\ \mu^2-1-3\mu+3\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \mu-1\\ \mu^2-3\mu+2\end{bmatrix}$$

Rank of coefficient matrix is 2

The system of eqn have solution only when the rank of Augmented matrix will be 2, and that can be possible if $\mu^2-3\mu+2=0 \Rightarrow \mu^2-2\mu-\mu+2=0 \Rightarrow \mu(\mu-2)-1(\mu-2)=0$

$(\mu-2)(\mu-1)=0 \Rightarrow \mu=1,2$

when $\mu=1$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \Rightarrow \begin{array}{l}x+y+z=1\\ y+3z=0\end{array}$$

i.e. if $z=c$

$y=-3c$

$x=1-c+3c=1+2c.$

$$\boxed{\begin{array}{l}x=1+2c\\ y=-3c\\ z=c\end{array}}$$

Illy for $\mu=2$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$x+y+z=1$

$y+3z=1$

$z=c$ ✓

$y=1-3c$ ✓

$x=1-1+3c-c$

$$\boxed{\begin{array}{l}x=2c\\ y=1-3c\\ z=c\end{array}}$$

Q. (b)

Apply rank test method to test the consistency and solve

$2x - y + 3z = 8,\ -x + 2y + z = 4,\ 3x + y - 4z = 0$

The system of eqn can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$R_3 + 3R_2,\ R_1 + 2R_2$
$$\begin{bmatrix} 0 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 7 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 12 \end{bmatrix}$$

R_{12}
$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 12 \end{bmatrix}$$

$3R_3$
$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 21 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 36 \end{bmatrix}$$

$R_3 - 7R_2$
$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ -76 \end{bmatrix}$$

Rank of Coefficient matrix = 3
Rank of Augmented matrix = 3
$\Rightarrow$ The system of eqn are consistent, hence unique solution.

$\therefore\ -x + 2y + z = 4$

$3y + 5z = 16$

$-38z = -76$

$\Rightarrow z = 2,\ y = 2,\ x = 2$ Ans.

Linear homogeneous equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Coefficient Matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- If the rank of $A = n$, then the eqns have trivial solutions. i.e. $x_1 = 0 = x_2 = \cdots = x_n$.
- If the rank of $A < n$, then the eqn have infinite solutions.

Ex Find the rank of A by reducing to triangular form by elementary row transformation.

Ex Does the following system of equations posses a non zero solution.

$$x + 2y - 3z = 0,\quad 2x - 3y + z = 0,\quad 4x - y - 2z = 0$$

Coeffts matrix
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 4 & -1 & -2 \end{bmatrix}$$

$$1(7) - 2(-8) - 3(10) = 7 - 16 - 30 = -39 \neq 0$$

$\rho(A) = 3$, $x = 0 = y = z$,

There is no non zero soln.

HA For what values of μ the equations $x + y + z = 1$, $x + 2y + 4z = \mu$, $x + 4y + 10z = \mu^2$ have a solution and solve them completely in each case.