Rank of a Matrix

A matrix is said to be a rank of r when
(i) it has atleast one non-zero minor of order r
(ii) Every minor of order higher than r vanishes.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = 1(40-42)-2(20-21)+3(12-12)=0$

3rd order Minor vanishes.

$\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 14-12 = 2 \neq 0$

2nd order minor exist

Rank of Matrix $A = 2$.

Also denoted as: $\rho(A) = 2$.

Elementary Transformation of a matrix

1. Interchange of any two rows (or column) : R_{ij} or C_{ij}
2. The multiplication of any row (or colum) by a non zero number : kR_i or kC_i
3. The addition to the elements of any row (or column) to the corresponding elements of any other row (or Column) multiplied by any number. $R_i + pR_j$ or $C_i + pC_j$

An elementary transformation is also called a row or Column transformation according as it applies to row or Column.

Note 1. Elementary transformations do not change either the order or the rank of a matrix

Equivalent Matrix: The matrices A & B are said to be equivalent if one can be obtained from other by a sequence of E-transformations. Two equivalent matrices have the same order as well as same rank.

$A \sim B$

Note • Rank of a unit matrix of order n is n

e.g. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\rho(B) = 3$

• Rank of null matrix is zero   e.g. $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\rho(C) = 0$

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ $|A| = 2 \neq 0 \Rightarrow 3^{rd}$ order minor exis

$\rho(A) = 3$

$D = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix} R_3 \to R_3 + R_1 \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\because$ it is not a null matrix

$\therefore \rho(D) = 1$

Eg Find the rank of the matrix

$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ operating $R_4 \to R_4 - (R_1 + R_2 + R_3)$

$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4th order Minor vanishes

$\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = -33 \neq 0 \Rightarrow \rho(A) = 3$.

4th order minor vanishes

Q/Find the rank of the matrix

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \underset{R_4-(R_1+R_2+R_3)}{\sim} \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_3-3R_2}{\overset{R_1-2R_2}{\sim}} \begin{bmatrix} 0 & +5 & 3 & 7 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3.$

$$\begin{bmatrix} 0 & 5 & 3 \\ 1 & -1 & 2 \\ 0 & 4 & 9 \end{bmatrix} = -1\,(33) = 33 \neq 0$$

eg. 1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 0 \end{bmatrix}$, $|A| = 1\,(-42) - 2\,(-21) + 3\,(0) = 0$

$\rho(A) = 2$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = +2 \neq 0$

HA 2) $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

3rd order minor vanishes

$\rho(A) = 2$

HA 3) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

4th order minor vanishes

$\rho(A) = 3$

Normal Form of a matrix

A non zero matrix A of order r can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a sequence of E-transformation is called Normal form

E1 Reduce the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

to a normal form and determine its rank.

$$C_2+C_1,\ C_3-2C_1,\ C_4+3C_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \overset{R_{24}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 4 & 5 & -8 & 14 \end{bmatrix}$$

$$R_4-4R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix} \quad C_4-2C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix} \quad \begin{matrix} R_4-5R_2 \\ R_3-3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

$$\sim R_4+2R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 0 \end{bmatrix} \sim -\tfrac{1}{8}R_4,\ -\tfrac{1}{2}R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim C_{34} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \rho(A) = 4$

HW find the rank of the matrix

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

by reducing it to Normal form.

Echelon form of a Matrix

i) Every row of the matrix A which has all its entries zero occurs below every row which has a non-zero entry

ii) The first non zero entry in each non-zero rows is equal to one (not necessary condition)

iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of matrix in Echelon form is equal to the number of non zero rows of the matrix

* Reduce the matrix to upper triangular matrix

$\rho(A) = 2$, i.e. number of non zero rows are 2.

eg

$$\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \overset{R_{13}}{\sim} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\overset{R_3-3R_1}{\sim} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \overset{R_{24}}{\sim} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix} R_3-4R_2 \\ R_4-2R_2 \end{matrix} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

$$\overset{R_4+2R_3}{\sim} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix} \overset{-\frac{1}{23}R_4}{\sim} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4$, i.e. number of non zero rows are 4