Linear dependence and independence

Vectors: Any quantity having n- Components is Called Vector.
- The Coefficient in a linear equation
  OR
- The element in a row or column matrix form a vector.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad X = [x_1, x_2 \cdots x_n]$$

Column vector — row vector.

- n number written in particular order, denote the vector X

Linear dependence: The vectors $X_1, X_2, \cdots X_n$ are said to be linearly dependent, if there exist r numbers $\lambda_1, \lambda_2, \cdots \lambda_r$, not all zero such that $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_r X_r = 0$

Linearly independent: The vectors $X_1, X_2, \cdots X_n$ are said to be linearly independent if $\exists$ r numbers $\lambda_1, \lambda_2, \cdots \lambda_r$ & if $\lambda_1 = 0, \lambda_2 = 0 = \cdots = \lambda_r = 0$ such that $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_r X_r = 0$

i.e. If no such numbers other than zero, exist, the vectors are said to be linearly independent.

Linear combination: $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_r X_r = 0$

$$\Rightarrow X_1 = -\frac{[\lambda_2 X_2 + \cdots + \lambda_r X_r]}{\lambda_1}$$

or $X_1 = \mu_2 X_2 + \cdots + \mu_r X_r$, then X_1 is said to be linear Combination of the vectors $X_2, X_3, \cdots, X_r$

Note: The number of linearly independent solutions of m homogeneous linear equations in n variables, $AX = 0$ is $(n-r)$, where r is the rank of the Coefficient Matrix e.g.1. Examine the following vectors for linear dependence.

$X_1 = (1,3,4,2)$, $X_2 = (3,-5,2,2)$, $X_3 = (2,-1,3,2)$

Solution : We know that $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

i.e $\lambda_1 (1,3,4,2) + \lambda_2 (3,-5,2,2) + \lambda_3 (2,-1,3,2) = 0$

is equivalent to

$\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$, $3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$,

$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$, $2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$

these are satisfied by $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -2$, which are not zero

$\therefore$ X_1, X_2, X_3 are linearly dependent vectors.

e.g.2 Examine the following vectors for linear dependence and find the relation if it exists.

$X_1 = (1,2,4)$, $X_2 = (2,-1,3)$, $X_3 = (0,1,2)$ and $X_4 = (-3,7,2)$

Soln: Let $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$

then $\lambda_1 (1,2,4) + \lambda_2 (2,-1,3) + \lambda_3 (0,1,2) + \lambda_4 (-3,7,2) = 0$

$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$.

$2\lambda_1 - 1\lambda_2 + 1\lambda_3 + 7\lambda_4 = 0$

$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$

we can write as

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

operating $R_2 - 2R_1$, $R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

operating $R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$, $-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$, $\lambda_3 + \lambda_4 = 0$

let $\lambda_4 = t \neq 0 \Rightarrow \lambda_3 = -t$, $\lambda_2 = 12t/5$, $\lambda_1 = -9t/5$

hence the vectors are linearly dependent

$\therefore$ from(i) $-\frac{9}{3} t X_1 + \frac{12t}{5} X_2 - t X_3 + t X_4 = 0 \Rightarrow 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

Ans

II.A Define linear dependence and independence of vectors. Examine for linear dependence of the following vectors,

$[1, 0, 2, 1]$, $[3, 1, 2, 1]$, $[4, 6, 2, -4]$, $[-6, 0, -3, -4]$

and find the relation between them if possible.