Basics Of Random Variable

- **Random Variable**(RV) is a real valued function defined on sample space.

➢ Every RV is a function, but every function is not a RV.

➢ A function is said to be a RV if it satisfies following two conditions:

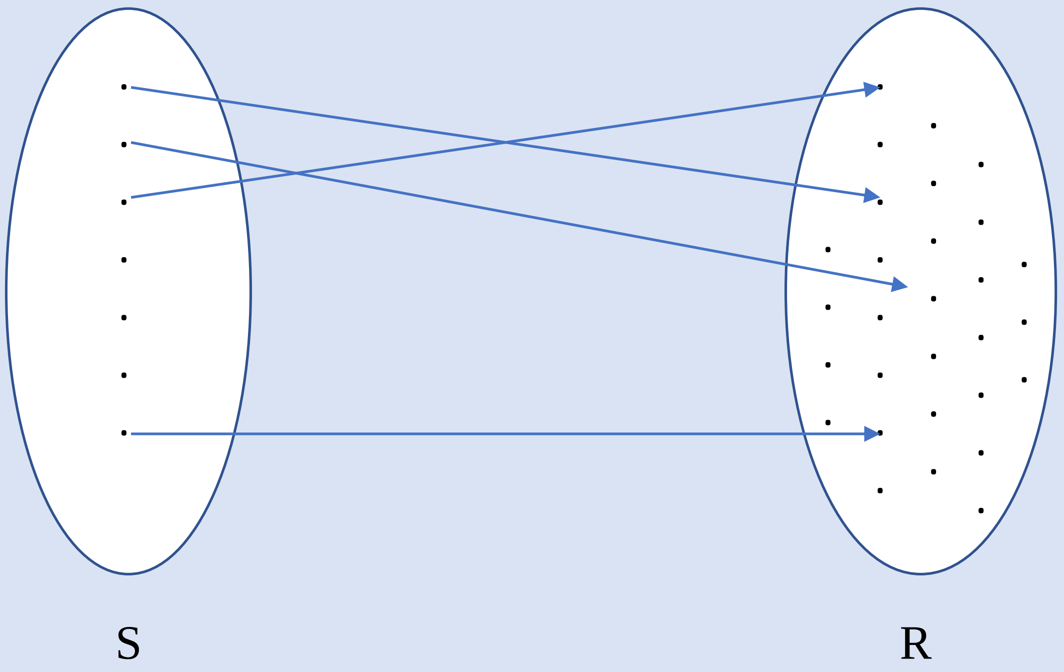
- **Condition 1:** The function must be defined on some sample space, i.e., the domain of the function must be some sample space.
- **Condition 2:** It should be a real valued function. i.e. the codomain of the function should be the set of real number.

$$f: S \rightarrow R$$

Notation of Random Variable& Its possible value

- Generally, the RVs are denoted by uppercases letters of English alphabets, viz, X,Y,Z etc. and their corresponding values are denoted by x, y, z etc respectively.
- If X is a R.V, then X: S→R

$$\therefore \forall s \in S, X(s) = x \in R$$



Examples of Random Variable.

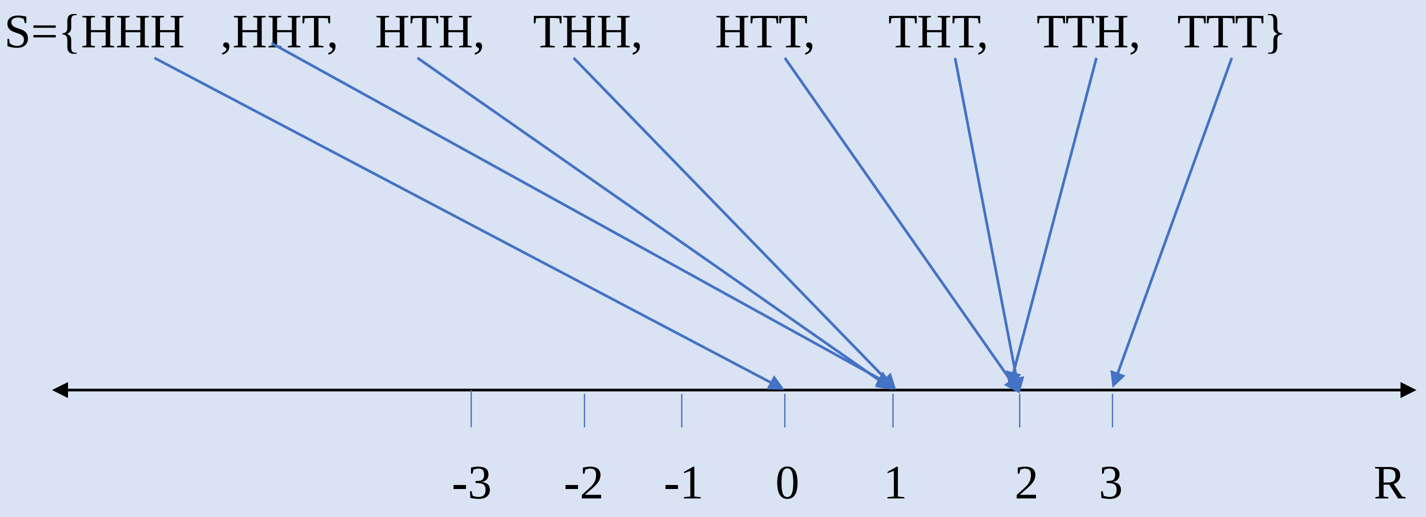
- Random Experiment: A coin is tossed three times.
- Sample Space: S ={HHH,HHT,HTH,THH,HTT,THT,TTH,TTT}
- Measurement on outcome: X Number of Tails(T) in the outcome.

X(HHH)=0, X(HTH)=1, X(HTT)=2, X(TTH)=2, X(HHT)=1,X(THH)=1,X(THT)=2,X(TTH)=2,X(TTT)=3,

Random variable(X)=

Number of tails in the

Outcome.



Some Important Observations About Of Random Variables

- In previous slide , every element of the sample space is mapped to exactly one real number.

$\therefore$ X is a function from S to R, i.e. $X: S \rightarrow R$

- ➢ An (informal)way of thinking is to regard a random variable as a measure about the outcome which takes a real number (e.g. how many heads occurs in a coin tossing experiment) or a measurement made on the outcome.
- ➢ For a given a random variable X on S, statement like " X=3" or "X≤ 3" are events. Specifically.
- ✓ "X=3" is the event $\{s \in s: X(s)=3\}$,i.e. the set of all outcomes in S for which X takes the value 3.
- ✓ "X≤ 3" is the event $\{s \in s: X(s) \leq 3\}$, i.e. the set of all outcomes in S for which X takes a value at most 3.

Discrete Random Variable

- Functions Of random variables: Any function you are likely to run across of a random variable or random variables is a random variable. So if and are random variables, then, and are all random variables.
- Spectrum Of R.V.: Range of a R.V. is known as spectrum of the R.V.

❑ Types Of Random Variables:-

1. Discreate Random Variables.
2. Continuous Random Variables.

1. Discreate Random variables:- X is a discrete random variable if:

- The set x of values of X is finite or countable .
- The Probability Mass Function(pmf) of X is a set of probability values p_i assigned to each of the values of x_i

➢ $f(x_i)=P(X = x_i), 0 \leq P(x) \leq 1.$

➢ $\sum p(x) = \sum P(X = x) = 1.$

Discrete Random Variable

- Cumulative Distribution Function: The cumulative distribution F(x) of a discrete random variable X with probability mass function f(x) is

$$F(X)= P(X \leq x)=\sum_{x_{i \leq x}} f(x_i)$$

The cumulative distribution of F(x) is an increasing step function with steps at the values taken by the random variable.

- Mean And Variance of discrete random variable:

We can summarize probability distribution by its mean and variance.

Mean or expected value(μ) is :

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

Variance of X is given as

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_{i=1}^{n} {x_i}^2 f(x_i) - \mu^2$$

Standard deviation of X is σ.

Example based on discrete random variable

- Question: Construct a probability distribution for drawing a card from a deck of 40 cards consisting of 10 cards numbered 1,10 cards numbered 3,and 5 cards numbered 4.

Solution:

Cards	**1**	**2**	**3**	**4**
Number of cards	10	10	15	5
Probability	0.25	0.25	0.375	0.125

Example based on discrete random variable

- By using above data to solve mean, variance, standard deviation to be find..

$$\text{Mean } (\mu) = \sum(x)[p(x)]$$

Substituting the value in above equation μ=11.25

$$\text{Variance: } \sigma^2 = E(X-\mu)^2 \text{ p(x).}$$

Substituting the value in variance above equation we get σ^2 =10.93

$$\text{Standard Deviation: } \sigma = \sqrt{\sigma^2} = \sqrt{\text{Variance}}$$

Substituting the value in the above equation σ=3.31

Continuous Random Variable

- Continuous random variable: A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. It is one that can assume any value over a continuous range of possibilities. Example: temperature, weight, electrical current, length ,pressure, time, voltage etc.

Probability Density Function(PDF)

For a continuous random variable X, a probability density function f(x) is a function such that,

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx=1$$

$$P(a \leq X \leq b)=\int_{a}^{b} f(x)dx$$ =area under f(x) from a to b for any a and b.

Continuous Random Variable

- Based on the definition, for a continuous random variable X and any value x, P(X=x)=0, because every point has zero width; i.e. zero area

However, in practice, when a particular x is observed, such as 14.47, this result can be interpreted as the rounded value that is actually in a range such as $14.465 \leq x \leq 14.475$

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

For the example, let the continuous random variable X denoted the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0,20mA], and assume that the probability density function of X is $f(x)=0.05$ for $0 \leq x \leq 20$.

1.what is the probability that a current measurement is less than 10 milliamperes?

$$P(X < 10) = \int_0^{10} f(x)dx = \int_0^{10} 0.05dx = 0.5$$

2.What is the probability that a current measurement is between 5 and 20 milliamperes?

$$P(5 < X < 20) = \int_5^{20} f(x)dx = \int_5^{20} 0.05dx = 0.75$$

Continuous Random Variable

Cumulative Distribution Function

The cumulative distribution function of a continuous random variable X, denoted as F(X), is

$$F(X)=P(X \leq x)=\int_{-\infty}^{x} f(u)\,du$$

$$\text{For } -\infty < x < \infty$$

The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating as long as the derivative exists.

$$f(x)=dF(x)/dx$$

Continuous Random Variable

Mean and Variance of a Continuous Random Variable

- The mean or expected value of the continuous random variable X with PDF f(x). denoted as μ or E(X), is

$$\mu=E(X)=\int_{-\infty}^{\infty} xf(x)dx$$

- The variance of X, denoted as $\sigma^2 = V(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty}(X-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx.\mu^2$

- The standard deviation of X , denoted as $\sigma = \sqrt{\sigma^2}$

Continuous Random Variable

- For the copper current measurement in the previous example, the mean of X is

$$\mu = E(X) = \int_0^{20} x f(x) dx = 0.05 \frac{x^2}{2} \int_0^{20} dx = 10mA.$$

$$\sigma^2 = V(X) = E(X-\mu)^2 = \int_0^{20} (x-10)^2 f(x) dx = \int_0^{20} x^2 f(x) dx - 100 = 33.33mA^2$$

Continuous Random Variable

The Expected value of a Function of a continuous Random Variable

Let X be a continuous random variable with PDF f(x),

$$E\big(g(X)\big) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

In the special case that $g(X) = aX + b$ for any constants a and b then the expected value of $g(x)$ is given by

$$E\big(g(X)\big) = aE(X) + b$$

Continuous Random Variable

Continuous Uniform Distribution

A random variable X has a continuous uniform distribution if

$$f(x) = \frac{1}{(b-a)}, \qquad a \le x \le b$$

Then,

$$\mu = E(X) = \frac{(a+b)}{2}$$

And

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

Continuous Random Variable

The mean and variance formulas can be applied with a=0 and b=20 in the previous example because

$$f(x) = \frac{1}{(b-a)} = \frac{1}{20} = 0.05 \qquad a \le x \le b$$

Therefore,

$$\mu = E(X) = \frac{(a+b)}{2} = 10mA$$

And

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12} = \frac{(20)^2}{12} = 33.33mA^2$$

Continuous Random Variable

Normal (Gaussian) Distribution

A random variable X with PDF

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty \le x \le \infty$$

is a normal random variable with parameters μ,

Where $-\infty \le \mu \le \infty$, and $\sigma > 0$, also

$$E(X) = \mu$$

and $V(X) = \sigma^2$

Continuous Random Variable

Standard Normal Distribution

The standard normal distribution is a special case of the normal distribution. It is the distribution that occurs when a normal random variable has a mean of zero and a standard deviation of one.

$E(X) = \mu$=0 and $V(X)$= $\sigma\lambda^2$=1

The normal random variable of a standard normal distribution is a called a standard score or z- score.

Continuous Random Variable

Normal Random Variable Standardization

Every normal random variable X can be transformed in to a z- score via the following equation:

$$Z = \frac{(X - \mu)}{\sigma}$$

Where X is a normal random variable, μ is the mean of X, and σ is the standard deviation of X.

Continuous Random Variable

Normal approximation to the binomial distribution

if X is a binomial random variable ,

$$Z = \frac{(X - np)}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable.

the approximation is good for:

$np > 5$ and $n(1-p) > 5$.

Continuous Random Variable

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable, with $E(X)=V(X)=\lambda$

$$Z = \frac{(X-\lambda)}{\sqrt{\lambda}}$$

is approximately a standard normal random variable .

the approximation is good for: $\lambda > 5$

Continuous Random Variable

Exponential Random Variable

The random variable X that equals the distance between successive counts of a Poisson process with mean $\lambda > 0$ is an exponential random variable with parameters λ. The probability density function of X is

$$f(X) = \lambda e^{-\lambda x} \quad for\ 0 \leq x < \infty$$

Also E(X)= μ=1/ λ and V(X)= σ^2=1/λ^2

Random Process

- Basic concept of Random Variable: in real life applications we are often interested in multiple observations of random values over a period of time. For example suppose that you are observing the stock price of a company over the next few months. In particular let S(t) be the stock price at time t $\epsilon[0, \infty)$. Here we assume t=0 refers to current time. Figure 1 shows a possible outcome of this random experiment from time t=0 to time t=1.

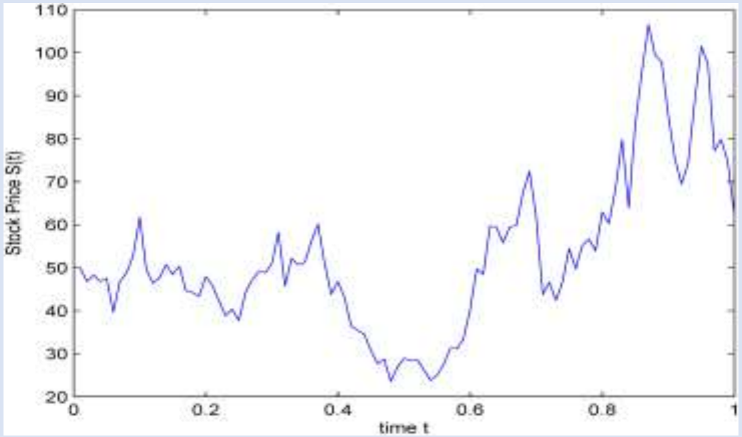


Figure 1. A possible realization of values of a stock observed as a function of time. Here, s(t) is an example of a random process.

Random Process

- Example:1 you have 100 dollars to put in an account with interest rate R, compounded annually. That is, if X_n is the value of the account at year n, then X_n =1000$(1+R)^n$, for n= 0,1,2,……….

The value of R is a random variable that is determined when you put the money in the bank, but it does not change after that. In particular, assume that R~ Uniform (0.04,0.05).

a. Find all possible sample functions for the random process {X_n, n=0,1,2,…..}.

b. Find the expected value of your account at year three. That is, find E[X_3].

Solution:

a. Here the randomness in X_n comes from the random variable R. as soon as you known R, you know the entire sequence X_n for n= 0,1,2,……… in particular, if R=r, then

$$X_n = 1000(1+r)^n, \text{ for all } n \in \{0,1,2,\ldots\}.$$

Thus, here sample functions are of the form f(n)= 1000$(1+r)^n$, n=0,1,2,…….., where r ∈[0.04,0.05]. You obtain a sample function for the random process X_n.

Random Process(contd..)

- (b) solution: The random variable X_3 is given by $X_3 = 1000(1+R)^3$.

if you let Y=1+R, then Y∼ Uniform (1.04,1.05), so

$$f_Y(y) = \begin{cases} 100 & 1.04 \le y \le 1.05 \\ 0, & \text{otherwise} \end{cases}$$

To obtain E[X_3], we can write

E[X_3]= 1000E[Y^3]

$= 1000\int_{1.04}^{1.05} 100\, y^3 dy$

= 1141.2

Random Process(contd..)

- Let $\{X(t), t\in [0,\infty)\}$be defined as

$X(t)= A+Bt$, for all $t \in [0,\infty)$,

Where A and B are independent normal N(1,1) random variables.

a Find all possible sample functions for this random process.

b Define the random variable Y= X(1). Find the PDF of Y.

c Let also Z= X(2). Find E[YZ].

Solution: (a) here we note that randomness in X(t) comes from the two random variables A and B. the random variable A can take any real value $a\in \mathbb{R}$. The random variable B can also take any real value $b \in \mathbb{R}$. As soon as we know that the values of A and B , the entire process X(t) is known. In particular, if A =a, B=b, then

$X(t)=a+bt$, for all $t \in [0,\infty)$,

Thus, here ,sample functions are of the form $f(t)=a+bt$,$t\geq 0$,where $a,b \in \mathbb{R}$. For any $a,b \in \mathbb{R}$ you obtain a sample function for the random process X(t).

Random Process(contd..)

- (b) we have Y=X(1)=A+B.

since A and B are independent N(1,1) random variables. Y=A+B is also normal with

EY=E[A+B]

=E[A]+E[B]

= 1+1=2.

We conclude that Y ~N(2,2): f_Y(y)= $\frac{1}{\sqrt{4*\pi}} e^{\frac{-(y-2)^2}{4}}$.

(c) we have E[YZ]= E[(A+B)(A+2B)]

= E[A^2+3AB+2B^2]

=E[A^2]+3E[AB]+2E[B^2]

= 2+ 3E[A]E[B]+2.2 =9.

Random Process (cont….)

- Note that at any fixed time $t_1 \in [0, \infty), S(t_1)$ is a random variable. Based on our knowledge of finance and the historical area, you might be able to provide a PDF for $S(t_1)$. If you choose another time $t_2 \in [0, \infty)$,you obtain another random variable $S(t_2)$ that could potentially have a different PDF. When we consider the value S(t) for t $\in [0, \infty)$ collectively, we say S(t) is a random process or a stochastic process. We may show this process by { S(t), t $\in [0, \infty)$ }.

Therefore, a random process is a collection of random variables usually indexed by time(or sometimes by space).

Types of Random process:

1. Continuous-time random process.
2. Discrete-time random process.

Random Process (cont.…)

- Continuous-time random process:- A continuous-time random process is a random process {X(t),t ∈ J}, Where J is an interval on the real line such as [-1,1],[0, ∞),(- ∞, ∞), etc. Example: the figure 1 is a continuous-time random process.
- Discrete-time random process:- A discrete-time random process (or a random sequence) is a random process { X(n)=X_n, n ∈ J} Where J is a countable set such as $\mathbb{N}$ *or* $\mathbb{Z}$.

A discrete-time random process is a process {X(t),t ∈ J},where J is a countable set. Since J is countable, we can write J={t_1, t_2 … … … … . .}. We can denote such a discrete- time process as {X(n), n=0,1,2…..}. Or, if the process is defined for all integers, then we may show the process by { X(n)=X_n, n ∈ $\mathbb{Z}$ }.

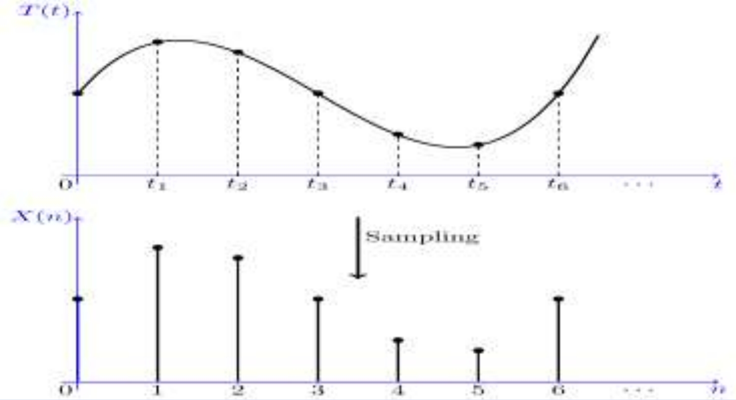


Figure 2. Possible realization of the random process {X(n), n=0,1,2…..}

PDF and CDF of Random Process

Consider the random process {X(t),t ∈ J}, for any t_0 ∈ J, X(t_0) is a random variable, so we can write its CDF

$F_{X(t_0)}$(x)=P(X(t_0)≤ x).

If t_1, t_2 ∈ J, then we can find the joint CDF of X(t_1) and X(t_2) by

$F_{X(t_1)X(t_2)}(x_{1,}x_2)$=P(X($t_1$)≤ x_1, X(t_2) ≤ x_2).

More generally for t_1, t_2,…………, t_n ∈ J, we can write

$F_{X(t_1)X(t_2)\ldots\ldots X(t_n)}(x_{1,}x_2,\ldots\ldots\ldots\ldots x_n)$=P(X($t_1$)≤ x_1, X(t_2) ≤ x_2, … … . . , X(t_n) ≤ x_n).

Similarly, we can write joint PDFs or PMFs depending on whether X(t) is continuous-valued or discrete random variables.

PDF and CDF of Random Process (Contd..)

- Example: Consider the random process $\{X_n$,n=0,1,2,…..}, in which X_i are i.i.d. standard normal random variables.

1. Write down f_{X_n}(x) for n=0,1,2,……
2. Write down $f_{X_m X_n(x_1,x_2)}$ for m≠ n…

Solution:-

1. Since $X_n \sim N(0,1)$, we have

$$f_{X_n}(x) = \frac{1}{\sqrt{2*\pi}} e^{\frac{-x^2}{2}}, \text{ for all } x \in R.$$

2. If m≠ n, then X_m *and* X_n are independent (because of the i.i.d. Assumption), so

$$f_{X_m X_n(x_1,x_2)} = f_{X_m}(x_1)\, f_{X_n}(x_2) = \frac{1}{\sqrt{2*\pi}} e^{\frac{-x_1^2}{2}} . \frac{1}{\sqrt{2*\pi}} e^{\frac{-x_2^2}{2}} = \frac{1}{2\pi}\exp\{-\frac{x_1^2+x_2^2}{2}\}, \text{ for all } x_1, x_2 \in R.$$

Mean and Correlations Functions

- Random process are collections of random variables. The expectation and variance were among the important statistics that we considered for random variable.
- Mean function of a Random Process:

For a random process {X(t),t ∈ J}, the mean function μ_X(t):J ⟶ R, is defined as

$$\mu_X(t) = E[X(t)]$$

The above definition is valid for both continuous-time and discrete-time random processes. In particular, if {X_n,n ∈ J} is a discrete-time random process ,then

$$\mu_X(n)=E[x_n], \text{ for all } n \in J.$$

The mean function gives us an idea about how the random process behaves on average as time evolves.

Mean and Correlations Functions(contd.)

For example if X(t) is the temperature in a certain city, the mean function μ_X(t) might look like the function show in figure below .the expected value of X(t) is lowest in the winter and highest in summer.

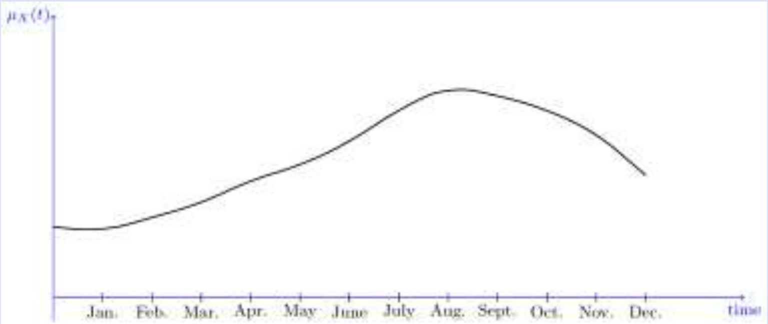


Figure: the mean function,$\mu_X(t)$, for the temperature in a certain city

Example:3 find the mean functions for the random processes given in Example 1 and 2.

Solution : for $\{X_n$,n=0,1,2,……..$\}$given in example1. we have

μ_X(n) = E[X_n]

=1000E[Y^n] (where Y=1+R ~ Uniform (1.04,1.05))

Mean and Correlations Functions(contd.)

μ_X(n) = E[X_n]

=1000E[Y^n] (where Y=1+R ~ Uniform (1.04,1.05))

= 1000$\int_{1.04}^{1.05} 100y^n$ dy

= $\frac{10^5}{n+1}$[$(1.05)^{n+1}$- $(1.04)^{n+1}$], for all n ∈ {0,1,2,……}.

For {X(t), t ∈ [0,∞)} given in example 2. we have

μ_X(t) = E[X(t)]

= E[A+Bt]

= E[A]+E[B]t

= 1+t, for all t ∈ [0,∞).

Autocorrelation and Autocovariance

- The mean function μ_X(t) gives us the expected value of X(t) at time t, but it does not give us any information about how X(t_1) and X(t_2) are related. To get some insight on the relation between X(t_1) and X(t_2) , we define correlation and covariance functions.

For a random process {X(t),t ∈ J}, the autocorrelation function or, simply the correlation function , $R_X(t_1, t_2)$, is defined by

$R_X(t_1, t_2)$=E[X(t_1) , X(t_2)], for t_1, t_2 ∈ J.

For a random process {X(t),t ∈ J}, the autocovariance function or simply the covariance function, $C_X(t_1, t_2)$,is defined by .

$C_X(t_1, t_2)$= cov(X(t_1) , X(t_2)) = $R_X(t_1, t_2)$- $\mu_X(t_1)$ $\mu_X(t_2)$, for t_1, t_2 ∈ J .

Note if we let consider $t_1 = t_2$=t, then

R_X(t, t)=E(X(t)X(t))

= E[X(t)^2], for t ∈ J.